Endomorphisms via Splittings

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Cryptography Seminar of Creach Labs

Thanks for the invitation! I hope you enjoy today's isojourney!

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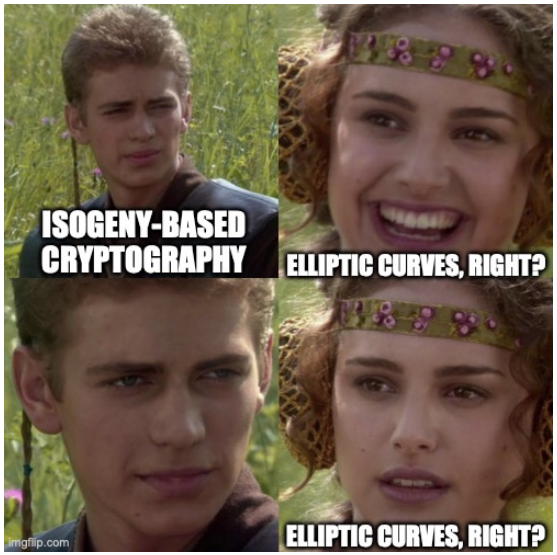
What does ‘dimension’ mean?



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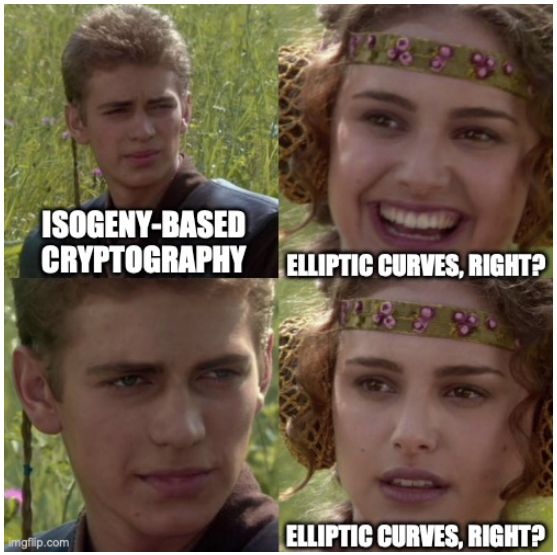

In isogeny-based cryptography, the main objects are the *principally polarized abelian varieties (PPAV)*.

One-dimensional PPAV: *elliptic curve*

Two-dimensional PPAV: *p.p. abelian surface (PPAS)*

— We have two flavors of PPAS over $\bar{\mathbb{F}}_q$: products of elliptic curves (reducible) and Jacobians of genus-2 curves (irreducible).

One-dimensional Assumptions

Two famous assumptions:

`Isogeny`: Given two supersingular elliptic curves E and E', find an isogeny $\varphi\colon E \to E'$

`EndRing`: Given a supersingular elliptic curve E, find the endomorphism ring $\mathrm{End}(E)$ of E.

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We now consider the following problem:

`OneEnd`: Given a supersingular elliptic curve E, find a non-constant endomorphism of E.

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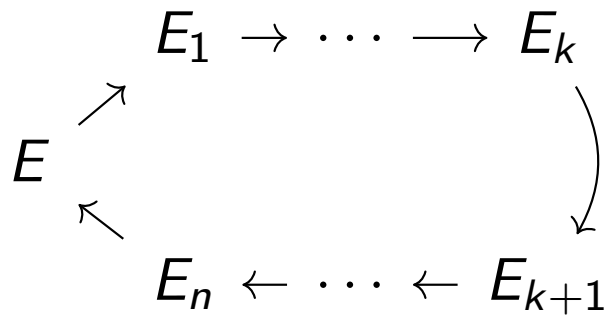

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The attack motivates the following problem.

`Splitting`: Given an irreducible PPAS A, find an isogeny $F \colon A \to E \times E'$.

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Bridges

Our main contribution is the reductions from OneEnd (and Isogeny) to the variants of Splitting.

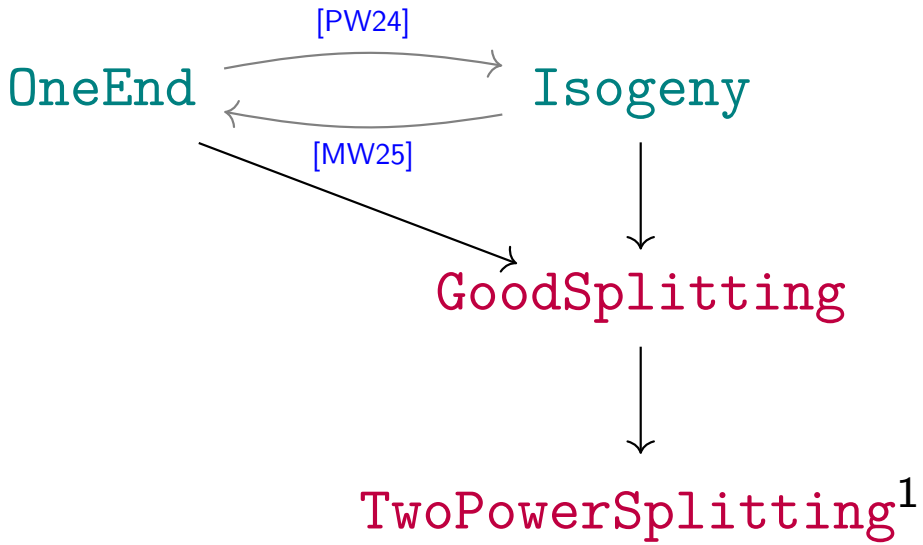

Figure: Our reductions

[1] It is essentially GoodSplitting with an additional degree restriction, making it much more practical to implement.

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Splitting is not only a (relatively) less-studied problem but also a very 'different' problem $\dashrightarrow$ new tools to solve it.[2]

[2] Tiny remark: There is also an ongoing Splitting-OneEnd project, which has a better running time, related to our framework.

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1. For an isogeny $f : A \to B$ between two PPAS A and B, there exists the ***adjoint isogeny*** $\tilde{f} : B \to A$ such that $f \circ \tilde{f} = [N]_A$ and $\tilde{f} \circ f = [N]_B$.

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4. The types:
   $E_1 \times E_2 \to E_3 \times E_4$: ***Product isogeny***
   $E_1 \times E_2 \to \mathrm{Jac}(C_1)$: ***Gluing isogeny***
   $\mathrm{Jac}(C_1) \to E_1 \times E_2$: ***Splitting isogeny***
   $\mathrm{Jac}(C_1) \to \mathrm{Jac}(C_2)$: ***Generic isogeny***

Good Isogenies

Given two $(2,2)$-isogenies $f_1 : A \to B$ and $f_2 \circ B \to C$:

1. $\ker(f_2 \circ f_1) = (\mathbb{Z}/2\mathbb{Z})^4 \Rightarrow (2,2,2,2)$-isogeny
2. $\ker(f_2 \circ f_1) = (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^2 \Rightarrow (4,2,2)$-isogeny
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Given an (N_1, N_1)-isogeny f_1 and an (N_2, N_2)-isogeny f_2, we say f_2 is a ***good extension*** of f_1 if $f_2 \circ f_1$ is an $(N_1 N_2, N_1 N_2)$-isogeny. Equivalently, $\ker(f_2) \cap \ker(\tilde{f_1}) = \{0\}$.

Kani's Lemma

Our usage of Kani's lemma:

An (N, N)-product isogeny $F : E \times E' \to E_1 \times E_2$ corresponds to the following commutative diagram (***isogeny diamond***).

$$\begin{array}{ccc} E & \xrightarrow{\phi_1} & E_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \psi_2} \\ E_2 & \xrightarrow{\psi_1} & E' \end{array}, \quad \deg \phi_i = \deg \psi_i = d_i, \ d_1 + d_2 = N$$

Moreover, the matrix form of F is

$$\begin{pmatrix} \phi_1 & \hat{\psi}_2 \\ -\phi_2 & \hat{\psi}_1 \end{pmatrix}$$

The Theta Coordinates[3]

We implement our algorithms in level-2 theta coordinates, so we informally introduce the necessary contents in the following few slides.

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• There are different level-2 theta structures on a single PPAV A: They are all related by *symplectic transformations*.

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The Product Theta Structure

Regarding a product $A := E \times E'$ of elliptic curves, we have a natural candidate of the theta structure — the ***product theta structure***, denoted by $\theta^A = \theta^E \times \theta^{E'}$:

$$\theta^A((P, P')) = (x_0x'_0 : x_0x'_1 : x_1x'_0 : x_1x'_1)$$

where $\theta^E(P) = (x_0 : x_1)$ and $\theta^{E'}(P') = (x'_0 : x'_1)$.

Furthermore, we explicitly know how to convert a non-product structure to the product one.

Hadamard Transform

We introduce an important symplectic transform — the ***Hadamard transform***.

It is defined as multiplication by a Hadamard matrix $\mathcal{H}_g$:

$$\mathcal{H}_g \in M_{2^g}, \quad \mathcal{H}_0 = \begin{pmatrix} 1 \end{pmatrix}, \quad \mathcal{H}_{i+1} = \begin{pmatrix} \mathcal{H}_i & \mathcal{H}_i \\ \mathcal{H}_i & -\mathcal{H}_i \end{pmatrix}$$

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$\tilde{\theta}^A = \mathcal{H} \circ \theta^A$ is called the ***dual theta structure*** of θ^A.

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Goal: evaluating a $(2,2)$-isogeny $f : A \to B$.

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Given $P \in A$, the formula is

$$\theta^B(f(P)) = \mathcal{H} \circ \mathcal{C}_{\tilde{b}} \circ \mathcal{H} \circ \mathcal{S}(\theta^A(P))$$

where $\mathcal{H}$ and $\mathcal{S}$ are known maps, $\tilde{b} = \mathcal{H}(\theta^B(0))$, and $\mathcal{C}_{\tilde{b}}$ means scaling by $(1/\tilde{b_0}, 1/\tilde{b_1}, 1/\tilde{b_2}, 1/\tilde{b_3})$.

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🔍 Rmk: We also have a similar formula for adjoint isogenies: $\theta^A(\tilde{f}(Q)) = \mathcal{C}_a \circ \mathcal{H} \circ \mathcal{S} \circ \mathcal{H}(\theta^B(Q))$.

Gluing Isogeny

The special case happens at gluing isogenies.

This problem was solved in [DMPR24] by assuming an additional knowledge: $\theta(P + T)$ where T is a 'suitable' 4-torsion point.

Main reason: transformation formula ensures that T swaps the coordinates, and therefore we can find the missing coordinate hidden by the vanishing coordinate.

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Note that the additional information is not trivial even if one has $\theta(T)$: we need $\theta(P)$, $\theta(T)$ and $\theta(P-T)$ to compute $\theta(P+T)$.

In our case, we can find this additional information efficiently, so we can evaluate all kinds of isogenies.

Random Walk — ThetaCGL ([KMM+25])

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where J is irreducible. We walk on the good $(2,2)$-isogeny graph by using *ThetaCGL hash function*. Very briefly, one could determine a good $(2,2)$-isogeny by choosing three bits.

Input: $\theta^A(0)$ and $s \in \{0,1\}^3$

Output: $\theta^B(0)$ where B is $(2,2)$-isogenous to A

The Lucky Moment

We know how to walk on the good $(2, 2)$-isogeny graph, but when could we stop?

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Lemma [Dup06]

Let $\left(a_{\binom{i}{j}}\right)$ be the even ($\langle i,j\rangle = 0$) level-4 theta null point of a p.p. abelian surface A. When exactly one a_i vanishes, A is reducible. Moreover, the corresponding level-2 theta structure is the product one if and only if $a_{\binom{11}{11}} = 0$.

We have a formula to calculate the squares of level-4 theta coordinates from level-2 theta coordinates.

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What does "suitable" mean?

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2. If we only consider the plain `Splitting` oracle, we might obtain some 'terrible' splittings, for example, the 'smartest' oracle:

$$\begin{pmatrix} [N] & 0 \\ 0 & [N] \end{pmatrix} : E \times E' \overset{f_1}{\underset{\tilde{f}_1}{\rightleftarrows}} \mathrm{Jac}(C)$$

3. In isogeny-based constructions, we like chains of good isogenies; otherwise, there might exist attacks, e.g., the collision attack [FT19] on the first 2-dimensional CGL hash function.

Goal

Recall that we know:

(1) The method to seek a good splitting.

(2) Evaluating a chain of good $(2,2)$-isogenies with a special care of the gluing.

$$E \times E' \longrightarrow \cdots \overset{\text{where}}{\longrightarrow} \mathrm{Jac}(C_1) \longrightarrow \cdots$$

It remains to locate the gluing.

Where is the gluing?

We can force the first isogeny to be a gluing by the following lemma.

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Lemma

Let $\theta^{E\times E'}(0) = (a_0 : a_1 : a_2 : a_3)$ be the theta null point of $E \times E'$ in the product structure. By applying the symplectic transformation

$$\sigma = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -i & i & -i & -i \\ 1 & -1 & -1 & -1 \\ -i & -i & i & -i \end{pmatrix}$$

the codomain from the radical formula is irreducible if one of the following holds:
(1) $E \not\cong E'$
(2) $j(E) = j(E') \notin \{0, 1728\}$ and $\theta^E(0) = \theta^{E'}(0)$.

Note that the automorphism σ is the same for any starting $E \times E'$.

$$E \times E' \xrightarrow{\sim} (E \times E')_\sigma \to A \to \cdots$$

Computational Reason

We use the following facts: Given $E \times E'$ and $f : E \times E' \to B$

(1) We exactly know the symplectic transformation converting the structure to the product one. More precisely, we know how to validly move the vanishing coordinates to $\binom{11}{11}$.

(2) The level-2 theta coordinates of dual null point of B is the first four coordinates of the level-4 coordinates of the theta null point of the domain.

So we can locate a gluing by constructing a special case: Using the converse of (1), we know the valid matrix σ converting $\binom{11}{11}$ to $\binom{00}{00}$.

If we are fortunate enough...

Assume we have found $F\colon E \times E' \to E_1 \times E_2$.

With the special care of the first step:
We can evaluate $F = \begin{pmatrix} \phi_1 & \hat{\psi}_2 \\ -\phi_2 & \hat{\psi}_1 \end{pmatrix}$ at a given $(P, Q) \in E \times E$ by the isogeny formula — especially at $(P, 0)$.
We can evaluate $\tilde{F} = \begin{pmatrix} \hat{\phi}_1 & -\phi_2 \\ \psi_2 & \psi_1 \end{pmatrix}$ at a given $(P', Q') \in E_1 \times E_2$ by the isogeny formula — especially at $(\phi_1(P), 0)$.

Therefore, we can extract $\psi_2 \circ \phi_1$.

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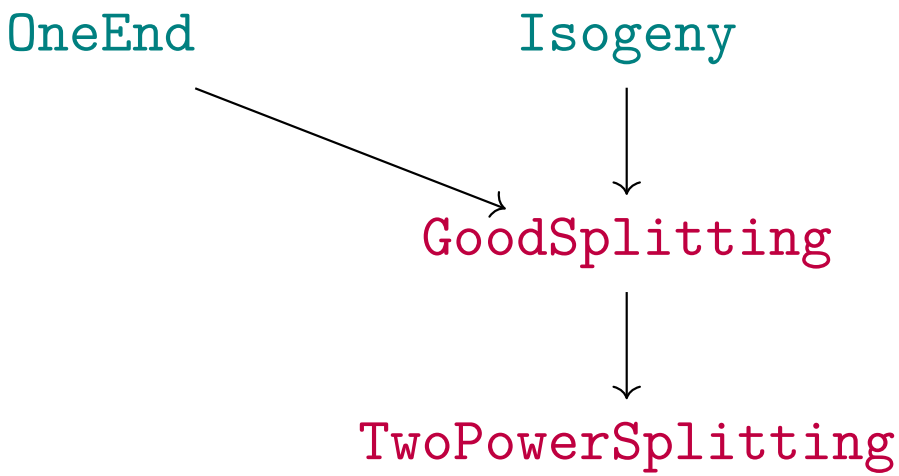
Therefore, we can extract $\psi_2 \circ \phi_1$.

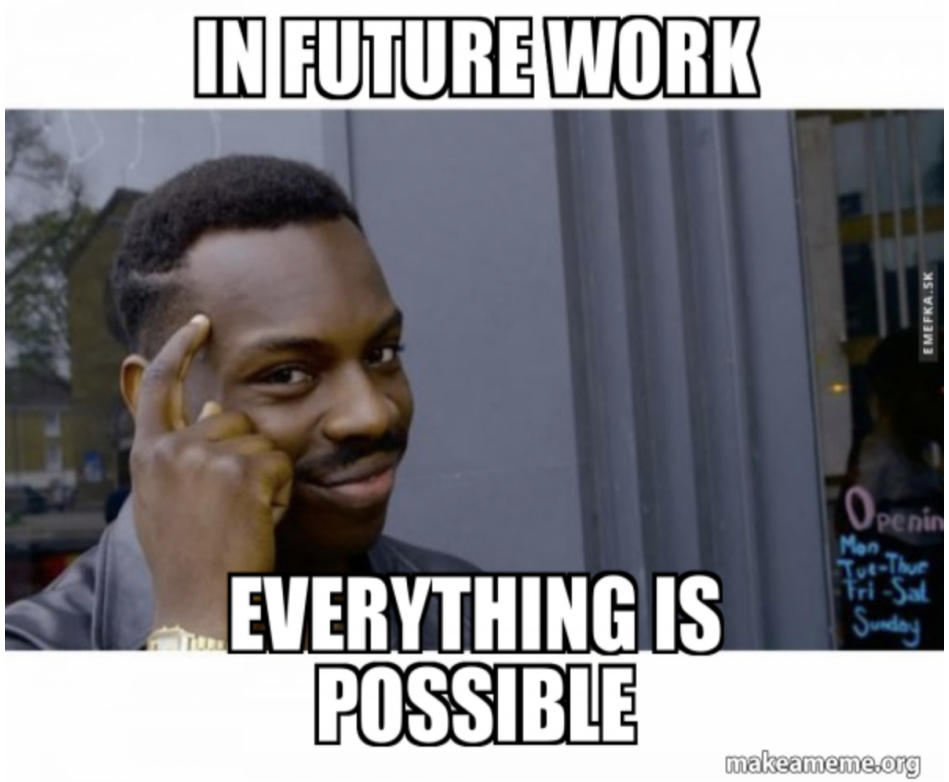
RMK: The degree of the extracted endomorphism (or isogeny) is of the form $\deg \phi_1(2^s - \deg \phi_1)$ where s is the length of the splitting. Note that it is usually non-smooth.

Reductions

We formally proved the reductions from `OneEnd` (resp. `Isogeny`) to `TwoPowerSplitting` and `GoodSplitting`: it remains to show the induced map is non-scalar (resp. non-zero).



Current/Future Works



1. Removing the 'good' restriction.
2. Connecting the splitting problems with other two-dimensional problems.
3. Studying the splitting problems themselves.

Finally, we find OneEnd — the ~~One~~End.

— A random photo I took, just to fill the void

Thank you!
Questions or Comments?

— ?

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Supplement: OneEnd to TwoPowerSplitting

Step 1: If the endomorphism $\psi_2 \circ \phi_1 : E \to E$ is a constant. By analyzing the matrix form and the kernels in detail, one can show that F is of the form $\begin{pmatrix} \phi_1 & \pm\phi_1 \\ -\phi_2 & \pm\phi_2 \end{pmatrix}$.

Step 2: Recall that an isogeny is defined by its kernel up to isomorphism. Consider $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) : E^2 \to E^2$, if a $(2,2)$-isogeny $f : E \times E \to A$ has kernel $\{(P,P) \mid P \in E[2]\}$, then $A \cong E \times E$. In particular, it's reducible.

Step 3: Assume $\psi_2 \circ \phi_1$ is a constant, then $\forall\, P \in E[2]$, $F((P,P)) = 0$ by Step 1. However, it means the first codomain is isomorphic to $E \times E$ which contradicts the fact that f_1 is a gluing.