Attacking the Supersingular Isogeny Problem:
From the Delfs–Galbraith algorithm to oriented graphs

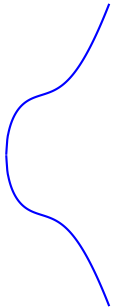
Arthur Herlédan Le Merdy

COSIC, KU Leuven

Cryptography seminar January 23, 2026, Rennes

Elliptic curves

Let p be a large prime.



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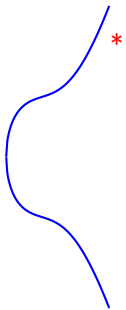

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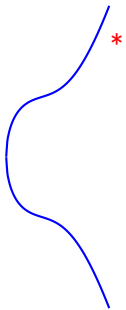

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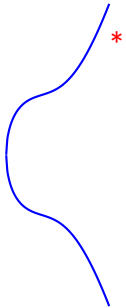

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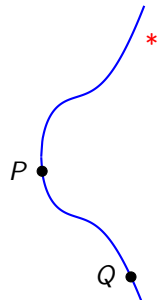

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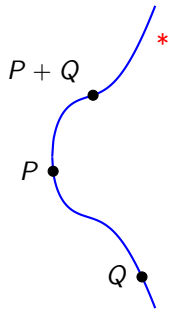

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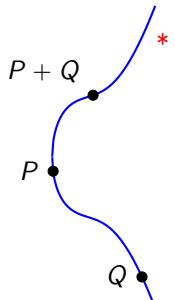

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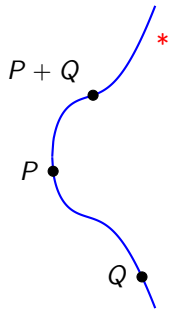

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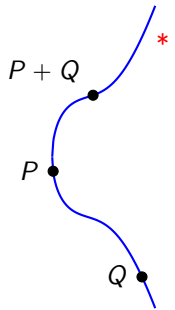

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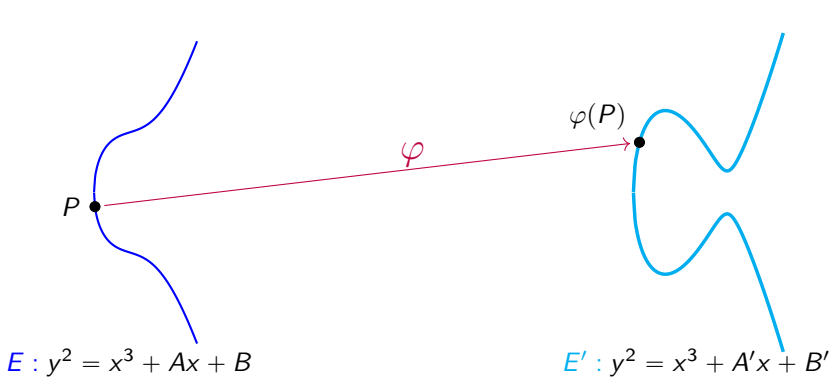
Post-quantum candidates: Lattice-based, Code-based,
Isogeny-based cryptography (Elliptic Curves come-back!)

Isogenies



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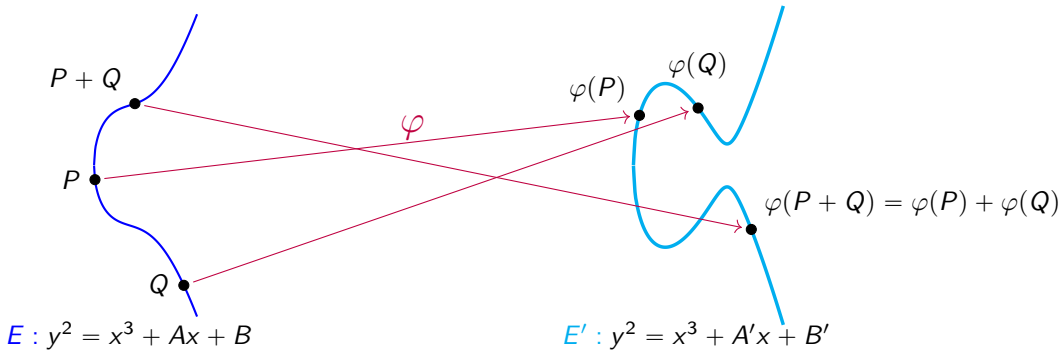

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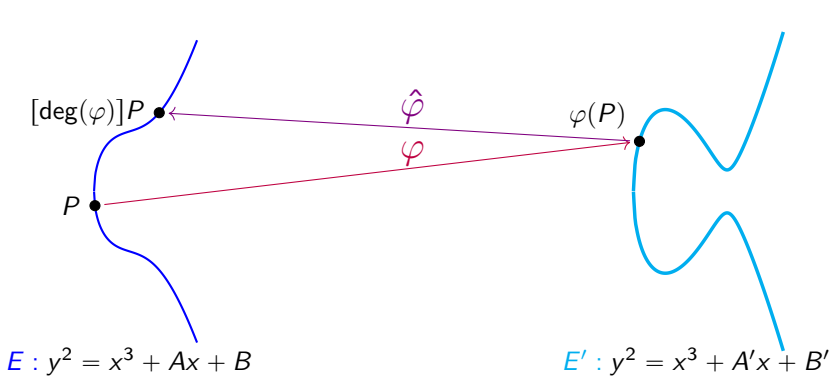

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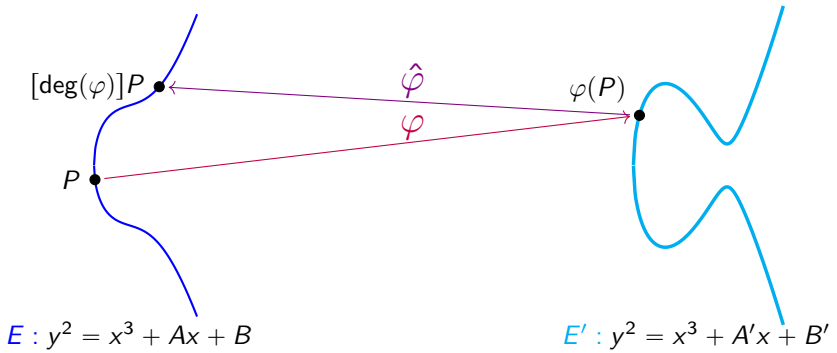

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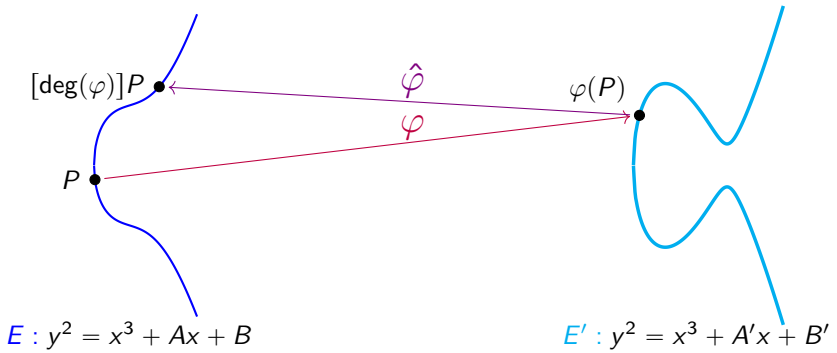

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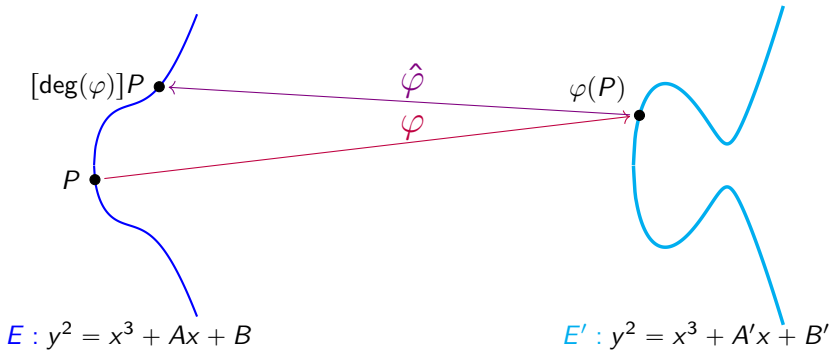

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CSIDH key exchange

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SQIsign digital signature

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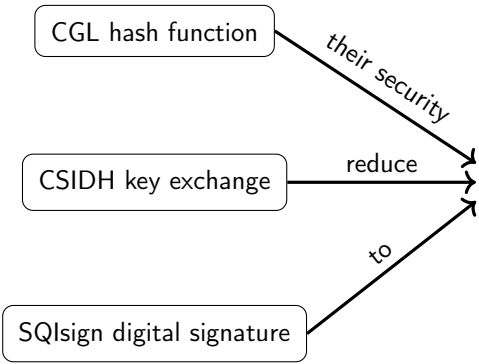

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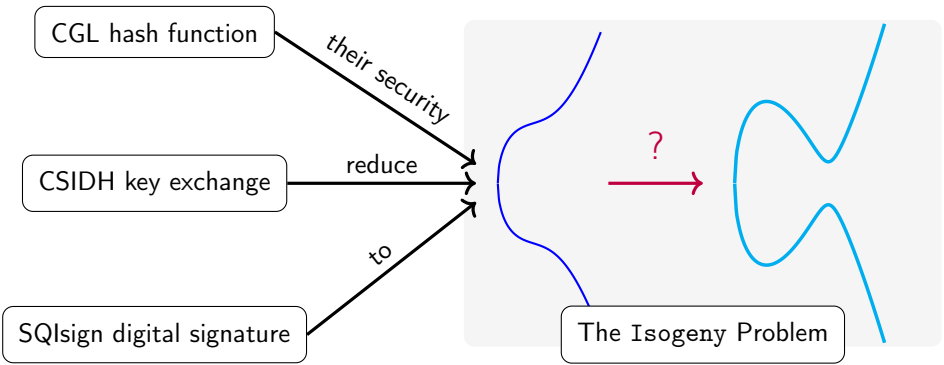

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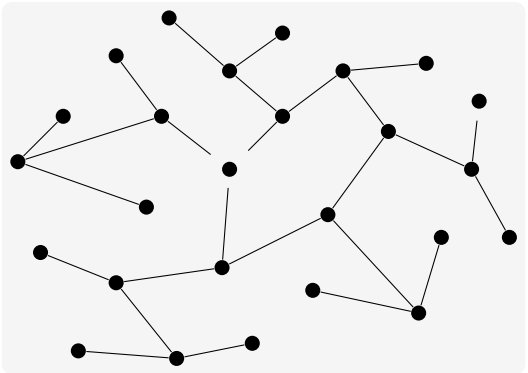

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vertices: supersingular elliptic curves up to **isomorphism**

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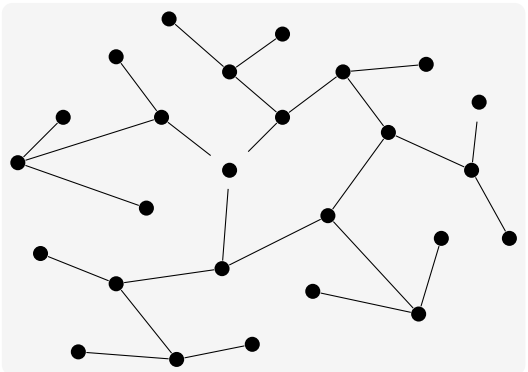

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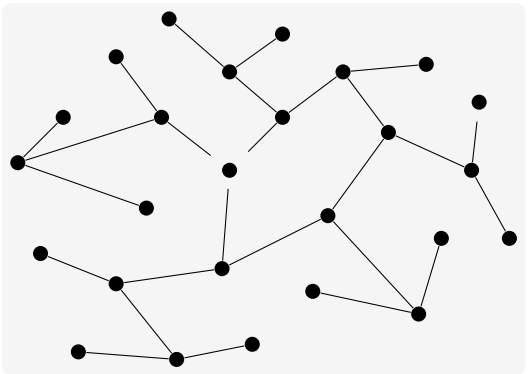

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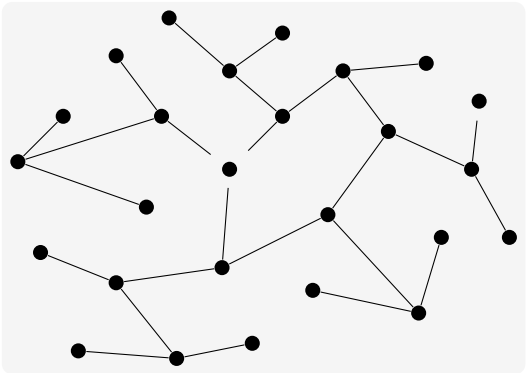

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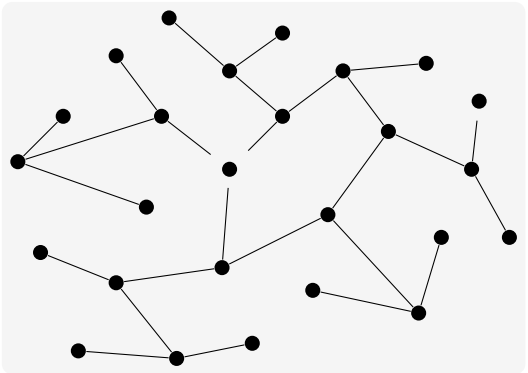

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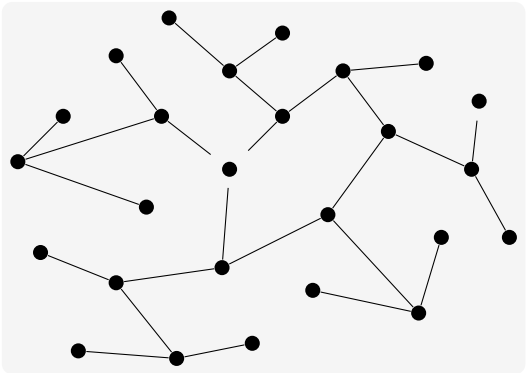

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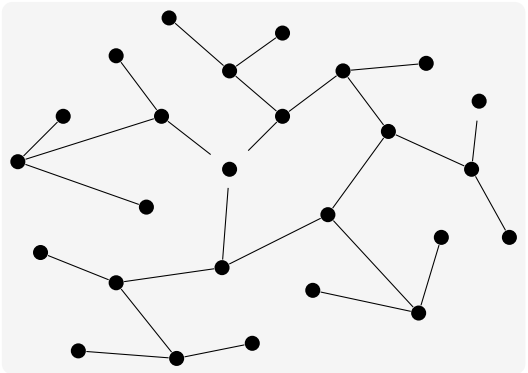

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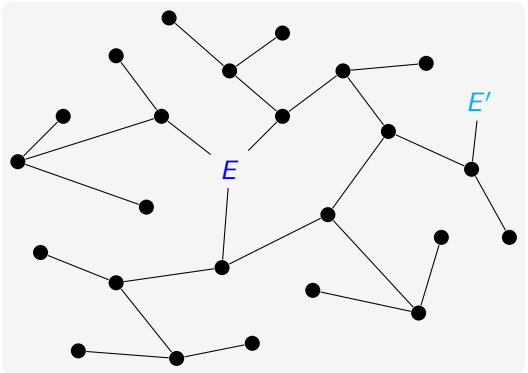

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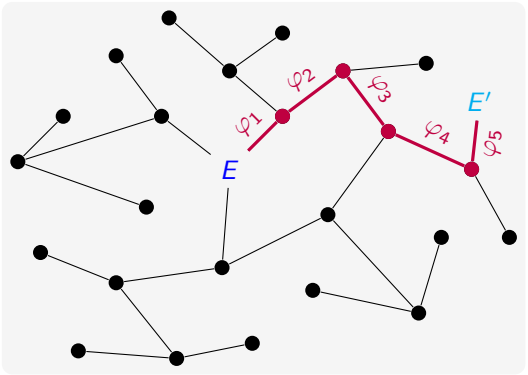

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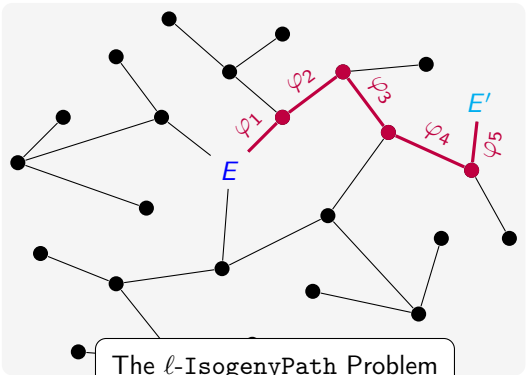

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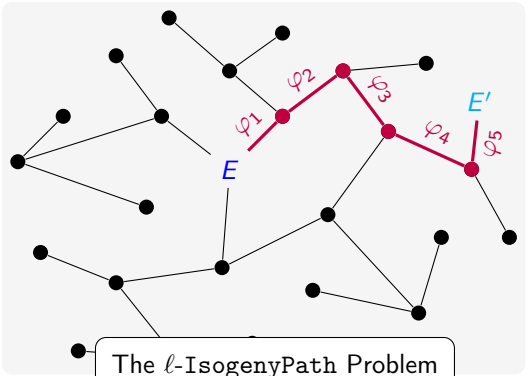


The ℓ-IsogenyPath Problem

⇕ [Wes22] under GRH

The Isogeny Problem

The ℓ-**isogeny graph** has

vertices: supersingular elliptic curves up to **isomorphism**

edges: isogenies of degree ℓ

Modular Polynomials

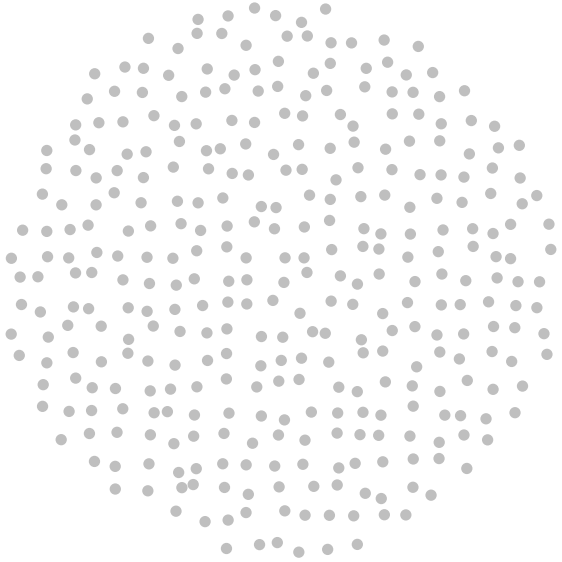
The modular polynomial $\Phi_\ell(X, Y)$ of level ℓ parametrizes ℓ-isogenies between elliptic curves:

$$E_1 \underset{\ell\text{-isogenous}}{\text{——}} E_2 \quad \text{iff} \quad \Phi_\ell(j(E_1), j(E_2)) = 0.$$

The ℓ-**isogeny graph** is

- $(\ell + 1)$-**regular**,
- **huge** ($\approx p/12$ vertices),
- **rapidly mixing**,
- **connected**.

The naive meet-in-the-middle approach



A (non-contractual) 2-isogeny graph over $\bar{\mathbb{F}}_p$.

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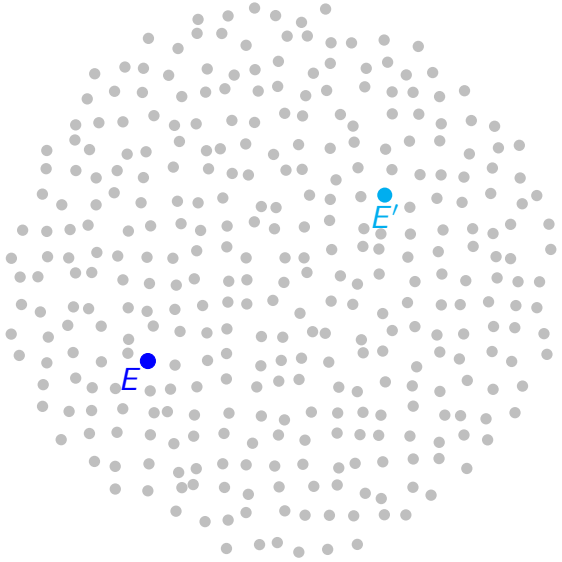

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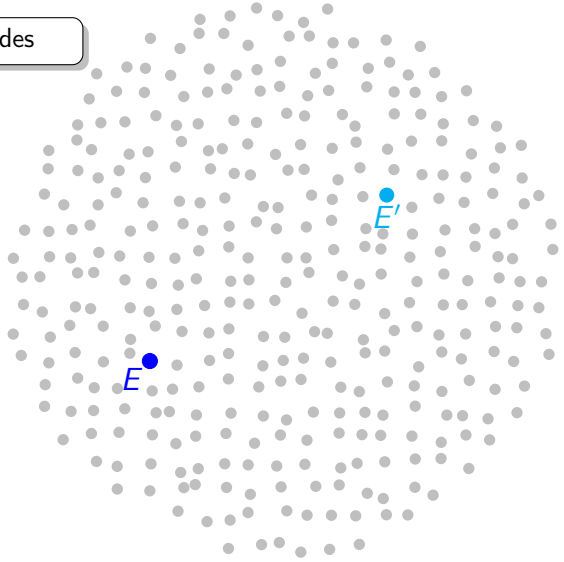

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Taking $O(\log p)$ random steps
almost gives
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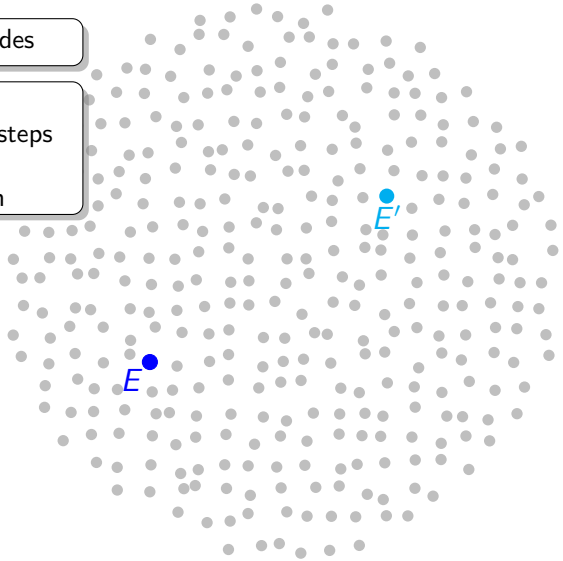

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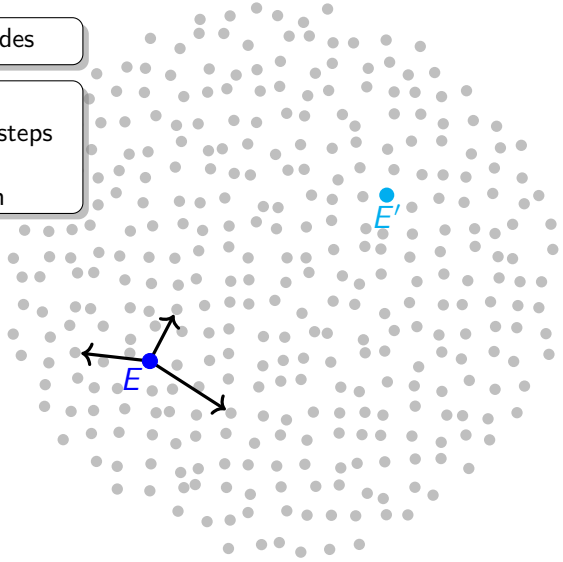

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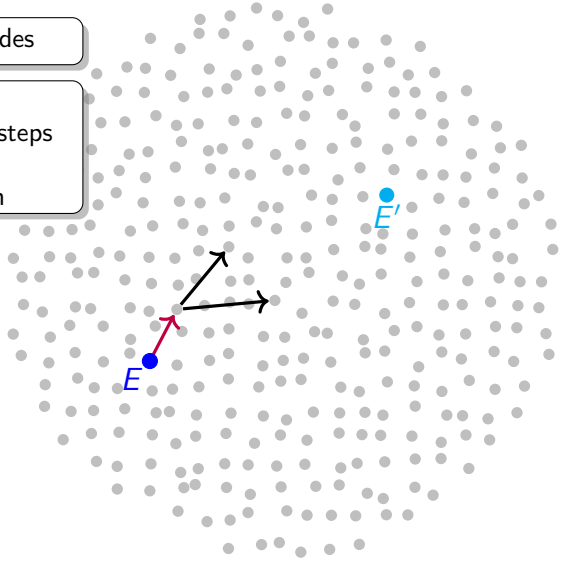

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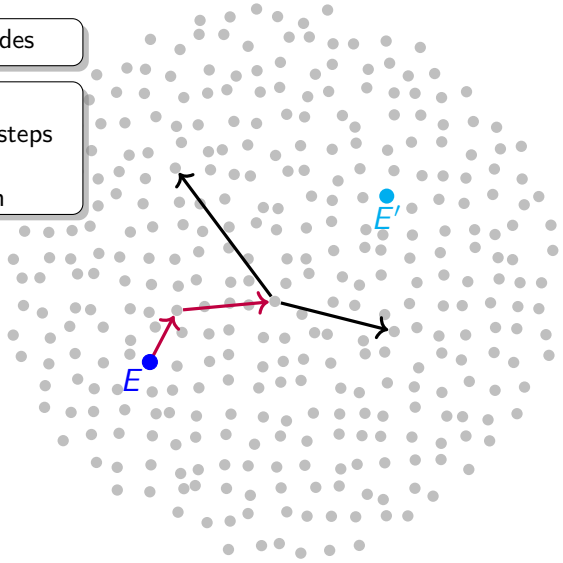

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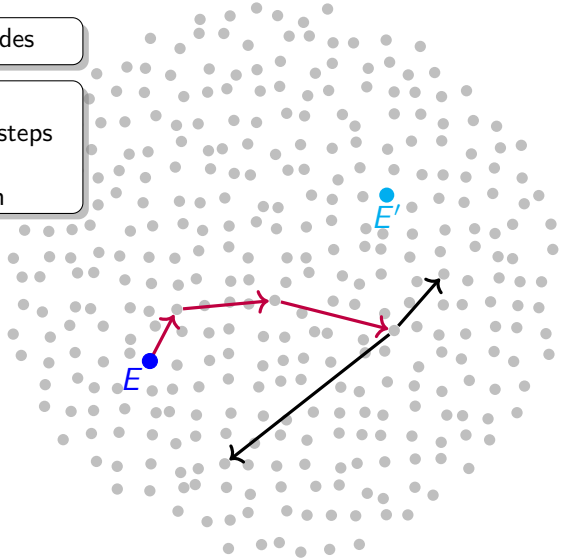

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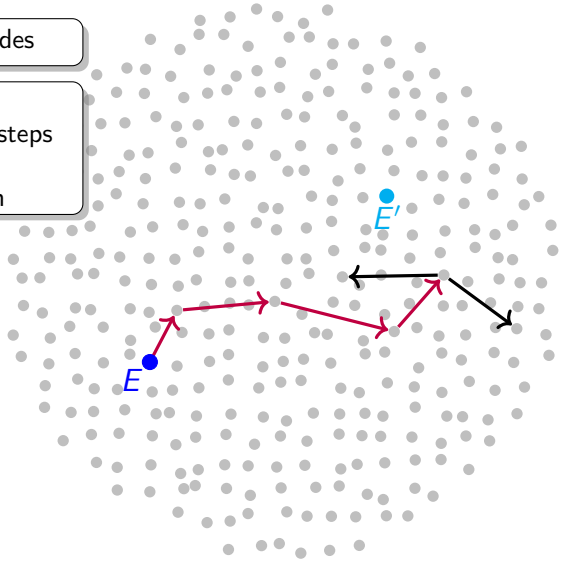

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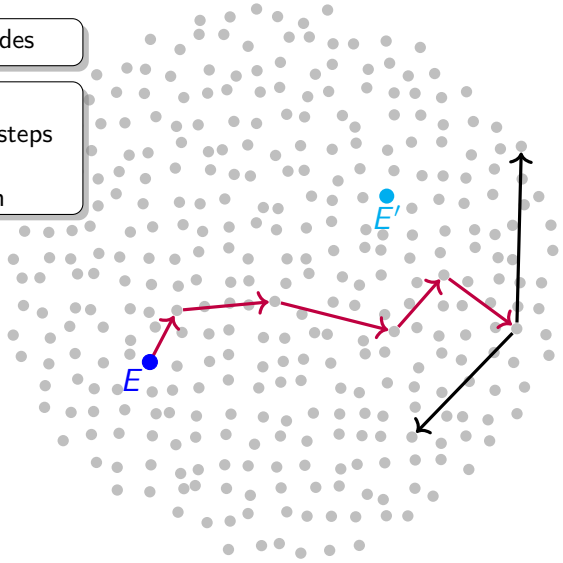

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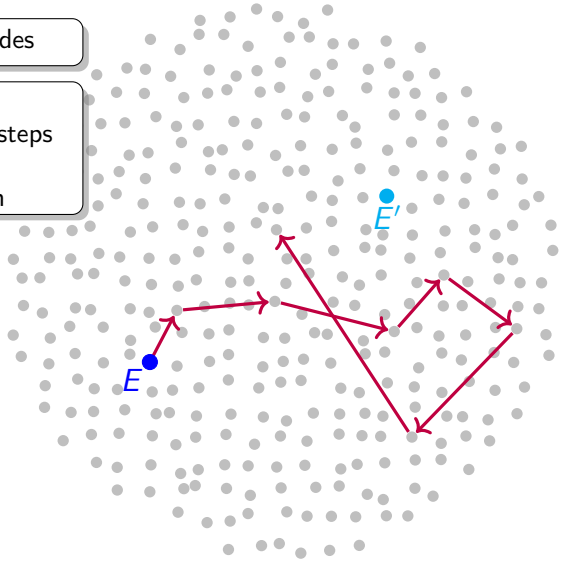

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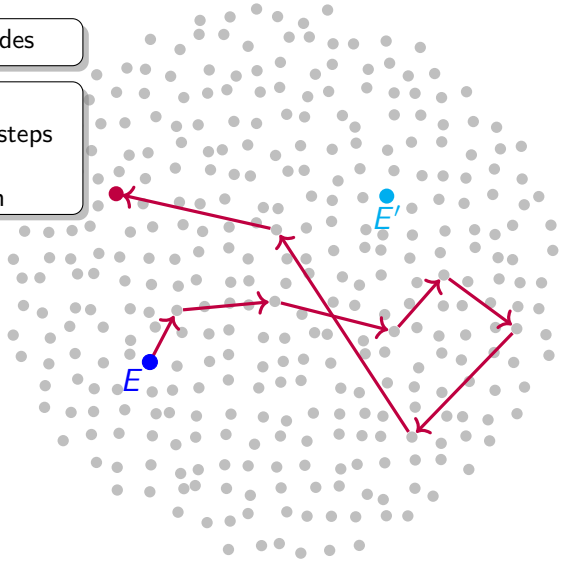

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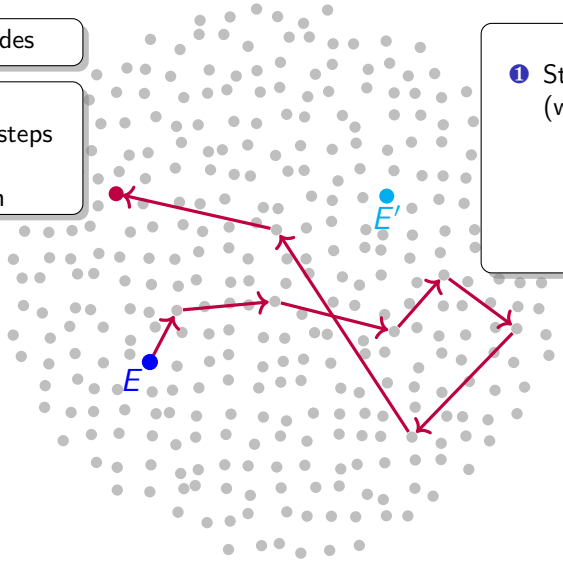

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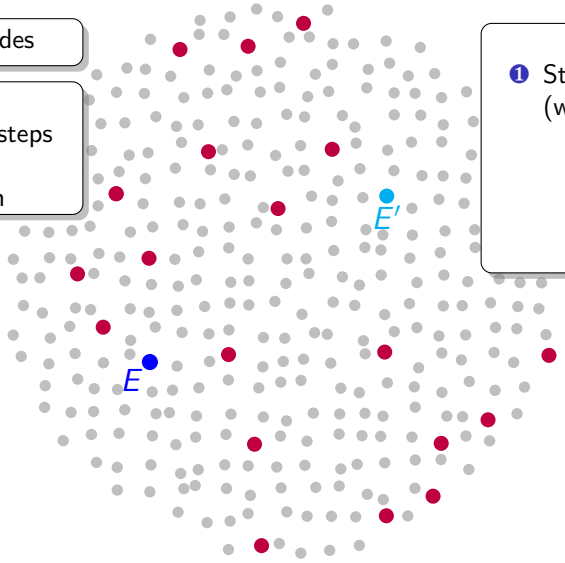

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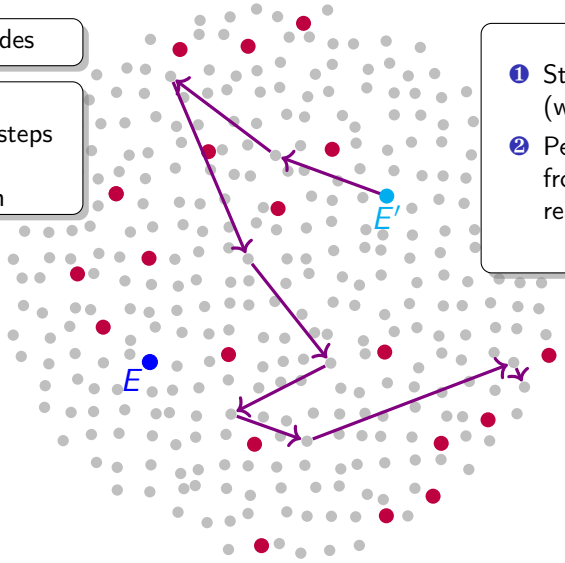

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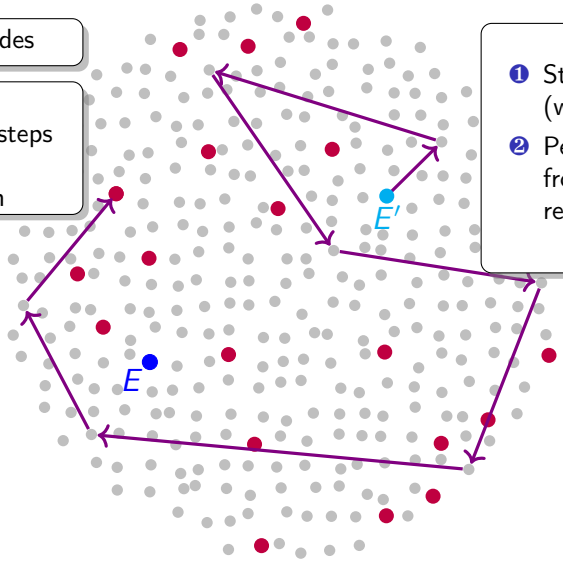

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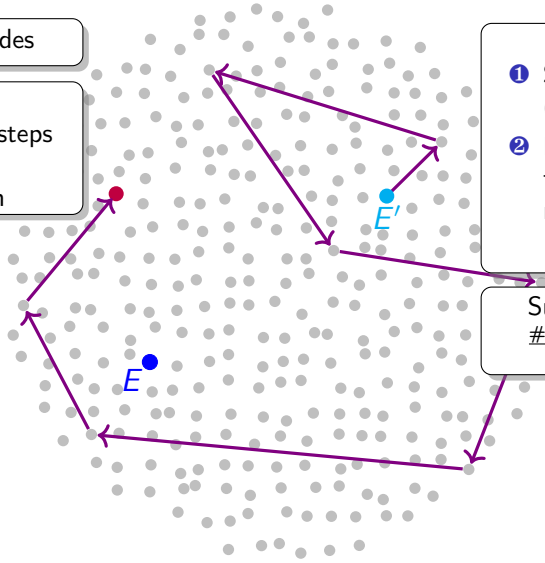

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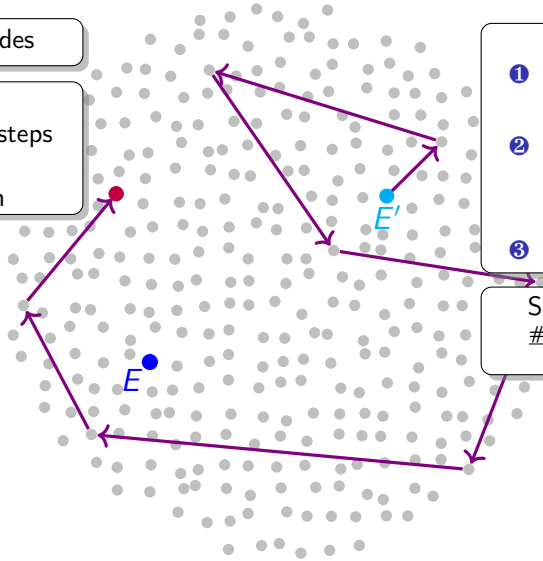

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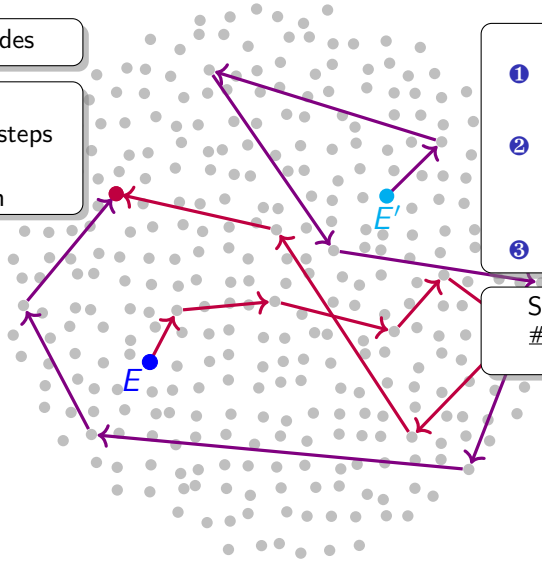

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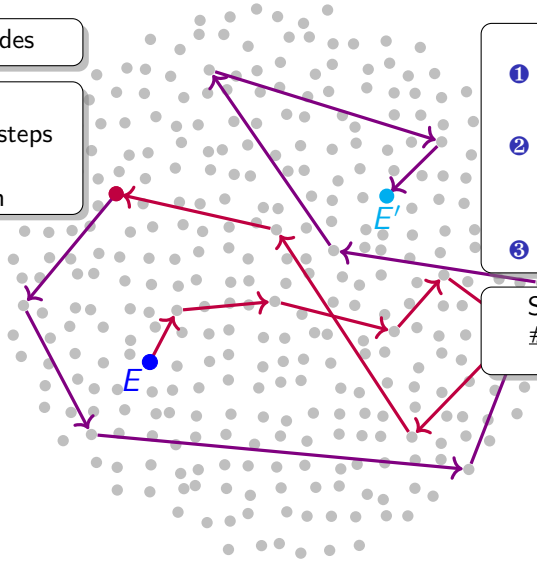

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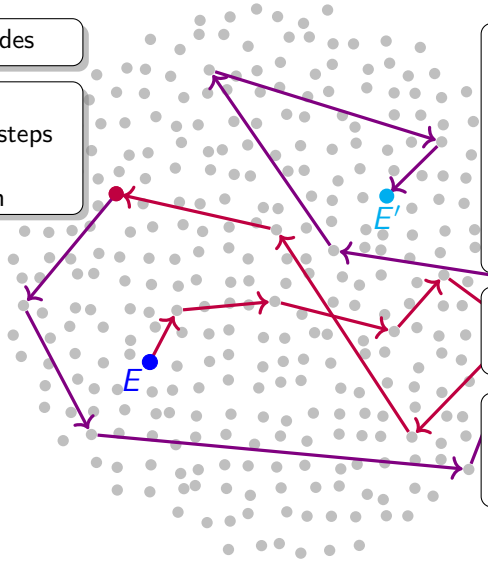

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Complexity:
Time: $\tilde{O}(\sqrt{p})$
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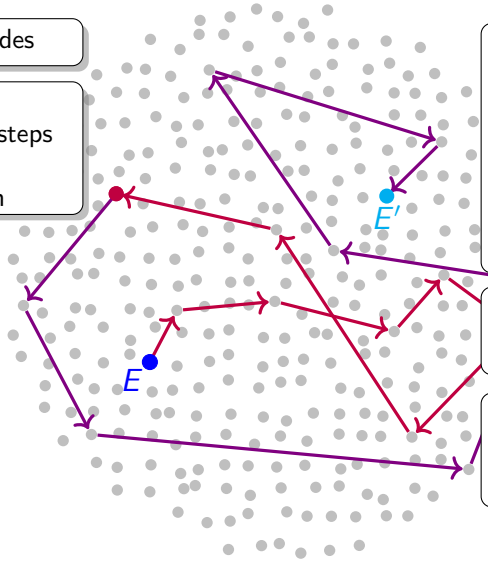

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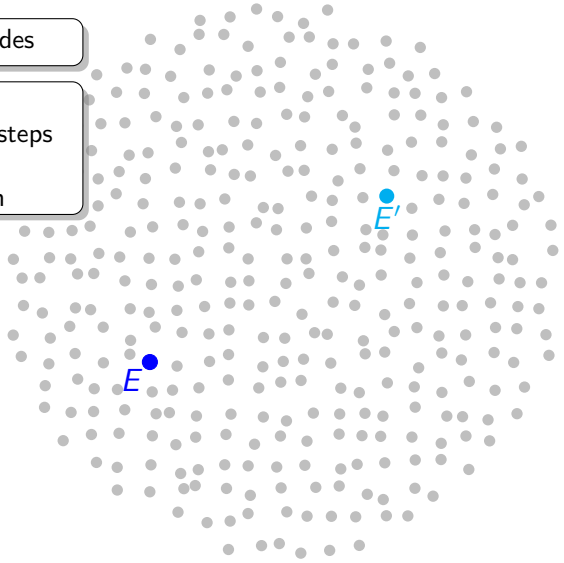

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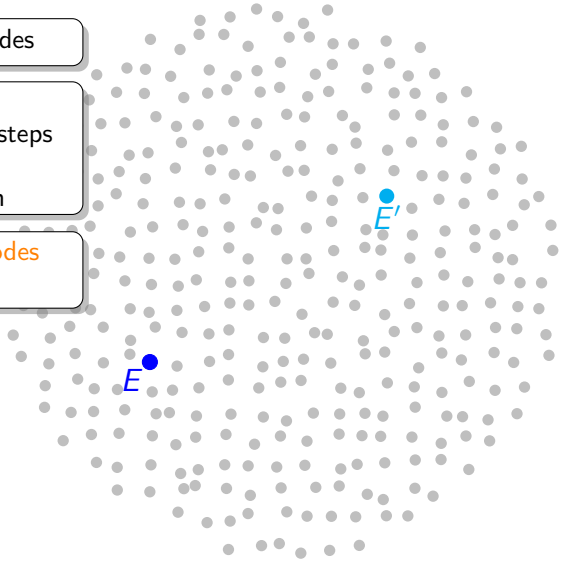

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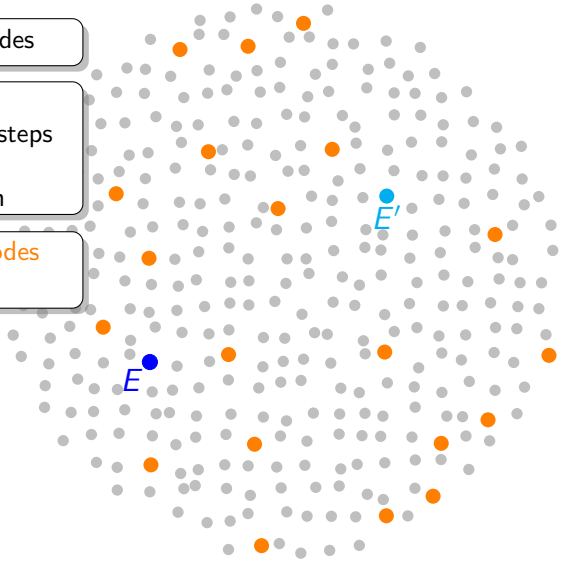

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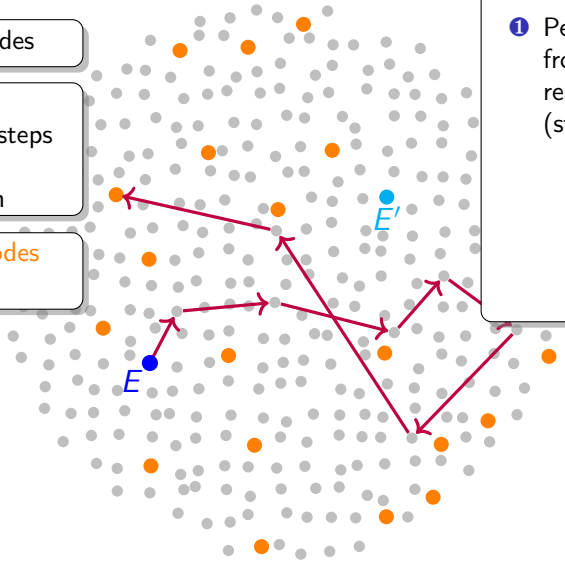

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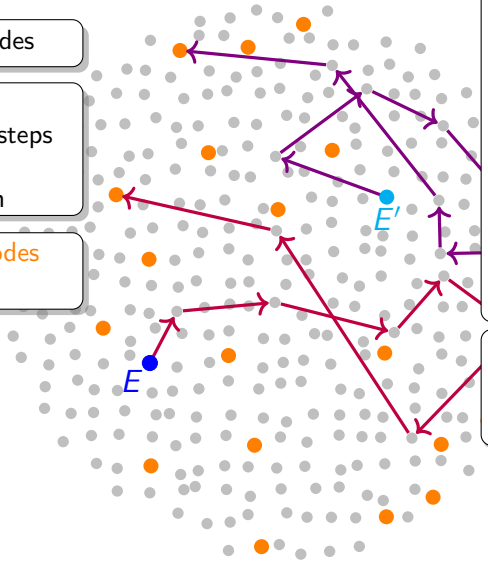

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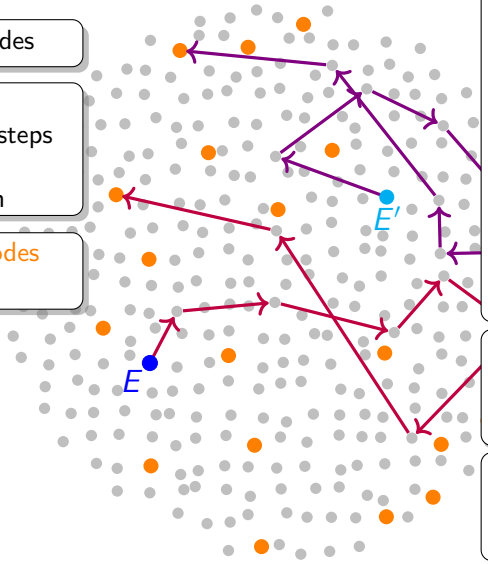

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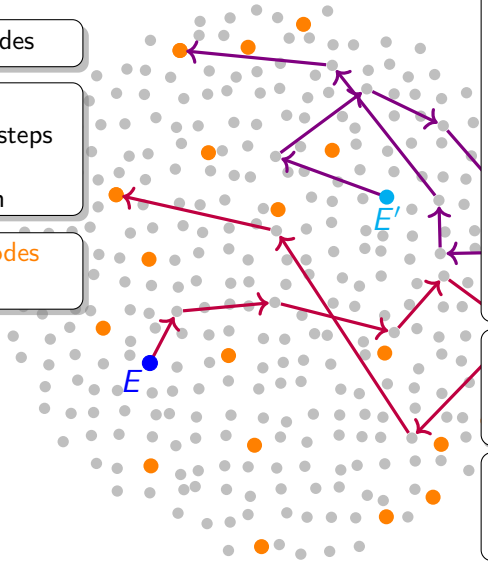

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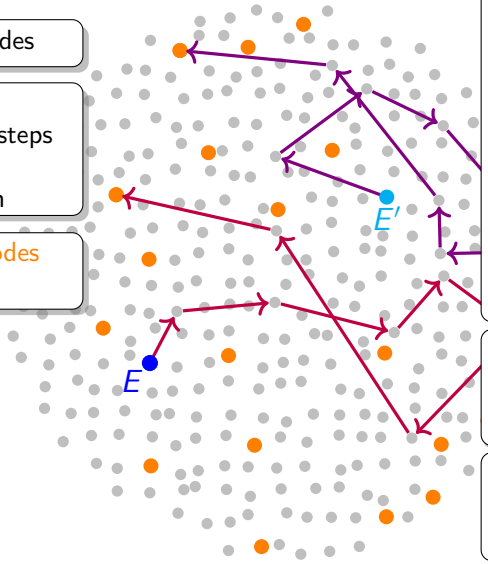

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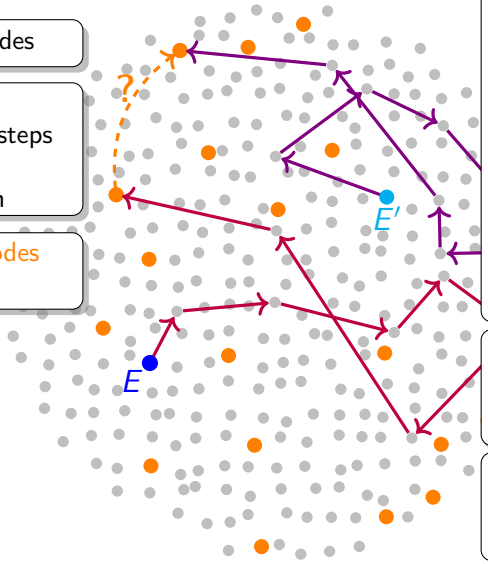

Method:

1. Perform random walks from E until reaching a curve over $\mathbb{F}_p$ (store the path)
2. Perform random walks from E' until reaching a curve over $\mathbb{F}_p$ (store the path)
3. Combine? the paths

Success probabilities for 1 and 2 are
$\frac{\#\text{ curves over } \mathbb{F}_p}{\#\text{ curves}} \approx \frac{\sqrt{p}}{p} = \frac{1}{\sqrt{p}}$

Complexity (so far):
Time: $\tilde{O}(\sqrt{p})$
Memory: $\text{polylog}(p)$

A (non-contractual) ℓ-isogeny graph over $\bar{\mathbb{F}}_p$.

The Delfs-Galbraith algorithm [DG16]

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Ramanujan graph:
Taking $O(\log p)$ random steps
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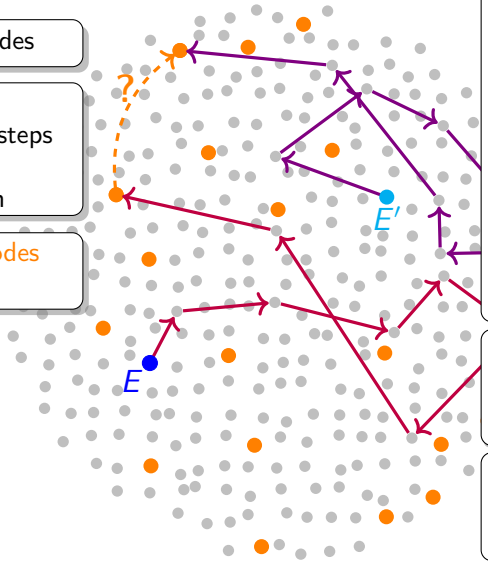

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1. Perform random walks from E until reaching a curve over $\mathbb{F}_p$ (store the path)
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Orientations

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$\alpha \in \mathrm{End}(E) \backslash \mathbb{Z}$ can be viewed as a **quadratic integer** with minimal polynomial

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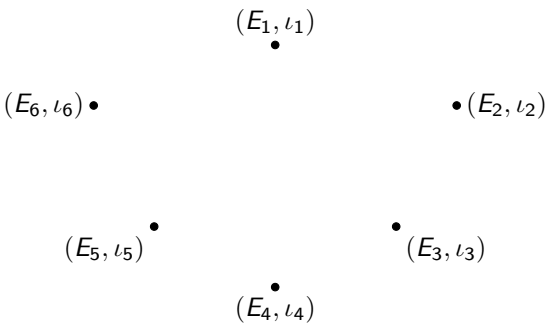
We say that ι is a **primitive orientation**,
if for any quadratic ring $\mathfrak{O} \supseteq \mathbb{Z}[\omega]$ such that $\mathfrak{O} \hookrightarrow \mathrm{End}(E)$, we have that $\mathfrak{O} = \mathbb{Z}[\omega]$.

Class group action

Let $SS_{\mathbb{Z}[\omega]}$ be the set of primitively $\mathbb{Z}[\omega]$-oriented elliptic curves, up to isomorphism.



Example when $|SS_{\mathbb{Z}[\omega]}| = 6$

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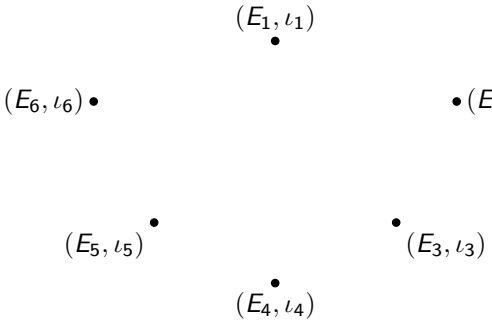

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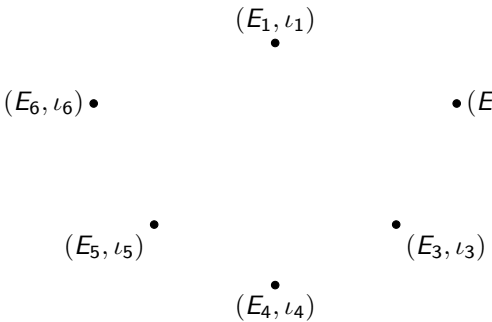

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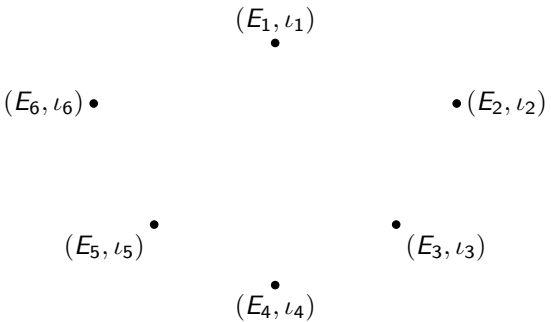

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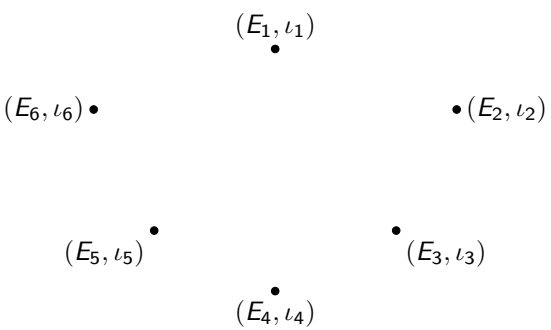

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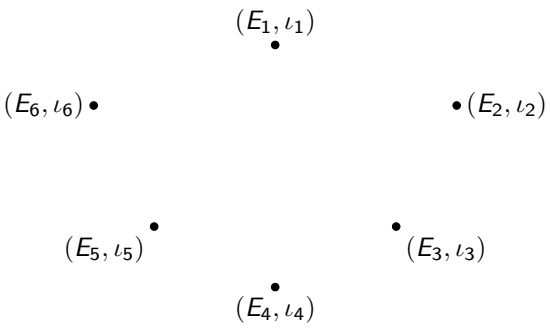

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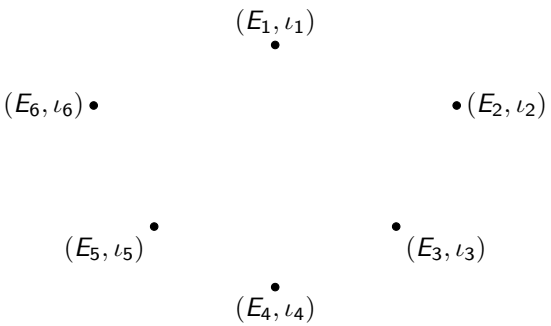

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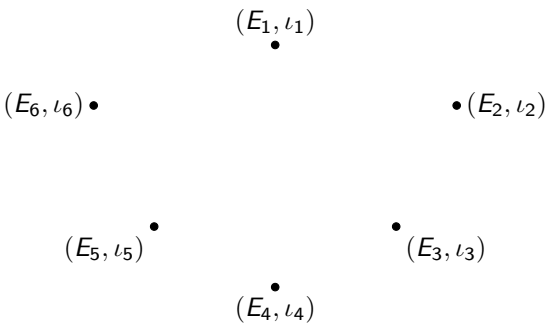

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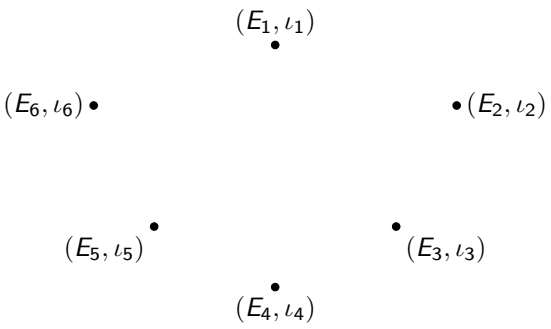

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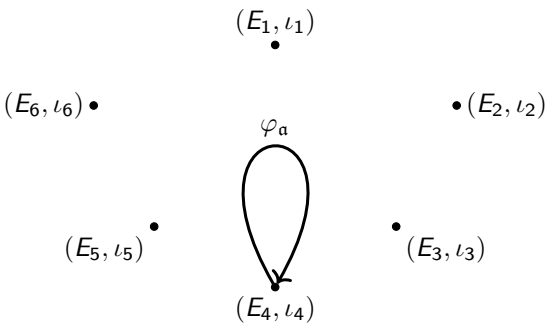

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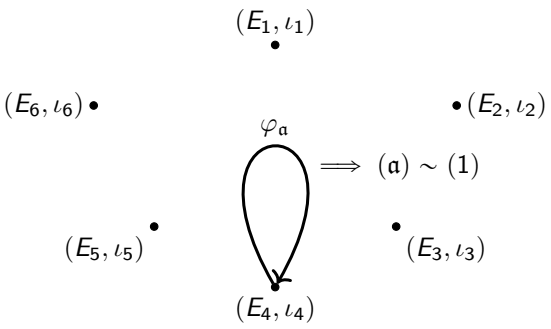

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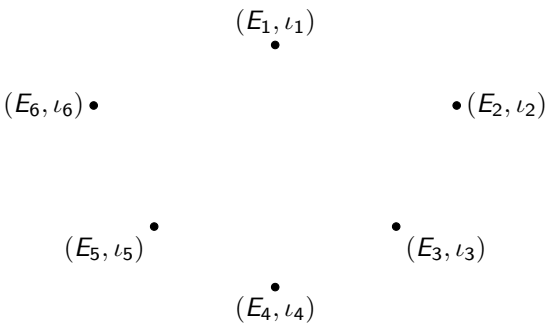

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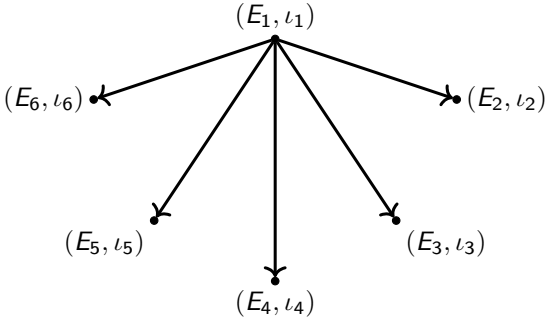

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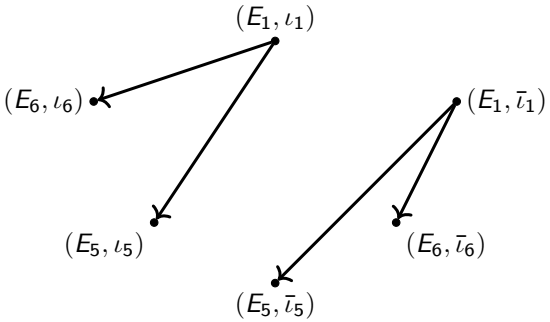

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Orientation over $\mathbb{F}_p$

Proposition [DG16]

If $p > 3$ then

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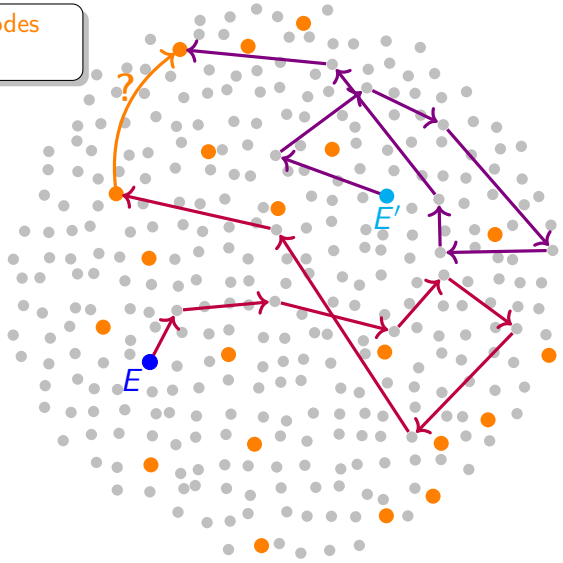
Corollary from [JMV09] (under GRH)

One can efficiently perform random walks over $SS_{\mathbb{Z}[\omega]}$ following a distribution close to uniform.

The Delfs-Galbraith algorithm (Step 2)

There are around $\sqrt{p}$ nodes defined over $\mathbb{F}_p$



A (non-contractual) ℓ-isogeny graph over $\bar{\mathbb{F}}_p$.

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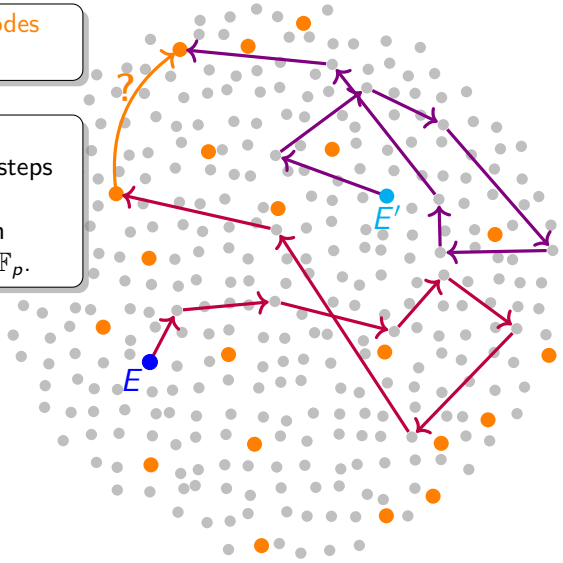
Taking $O(\log p)$ random steps *almost* gives a uniform distribution on curves defined over $\mathbb{F}_p$.



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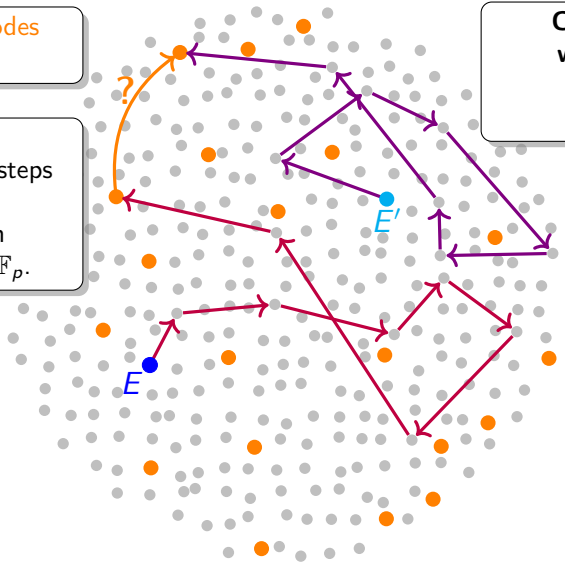

Complexity of Step 2 with a naive MITM:
Time: $\tilde{O}(p^{1/4})$
Memory: $\tilde{O}(p^{1/4})$

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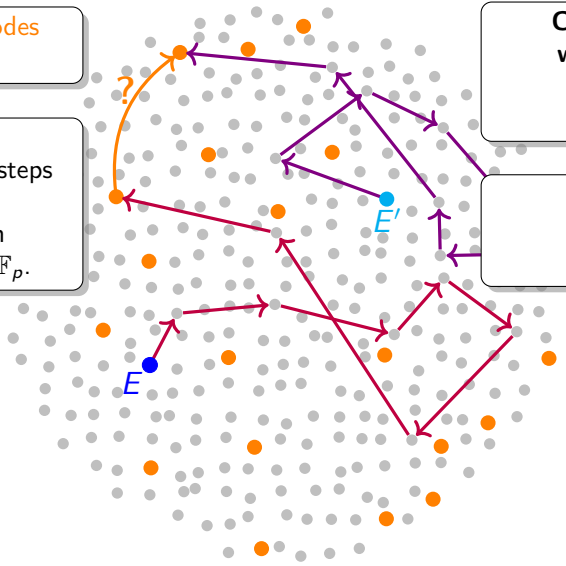

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Time: $\tilde{O}(p^{1/4})$
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Total complexity:
Time: $\tilde{O}(\sqrt{p})$
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A (non-contractual) ℓ-isogeny graph over $\bar{\mathbb{F}}_p$.

The Delfs-Galbraith algorithm (Step 2)

There are around $\sqrt{p}$ nodes defined over $\mathbb{F}_p$

Expander graph:
Taking $O(\log p)$ random steps
almost gives
a uniform distribution
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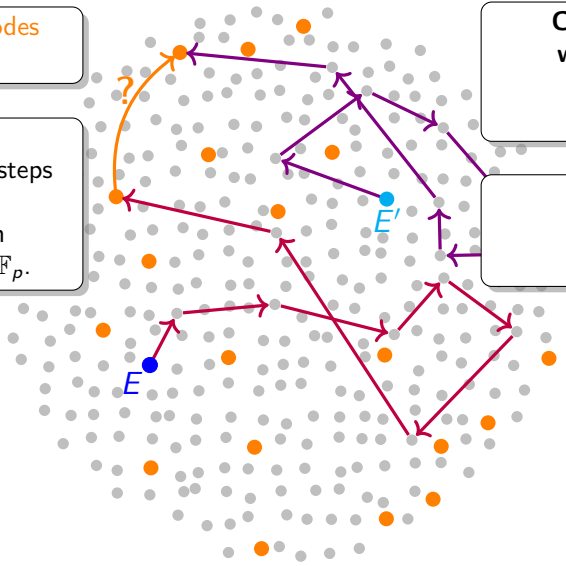

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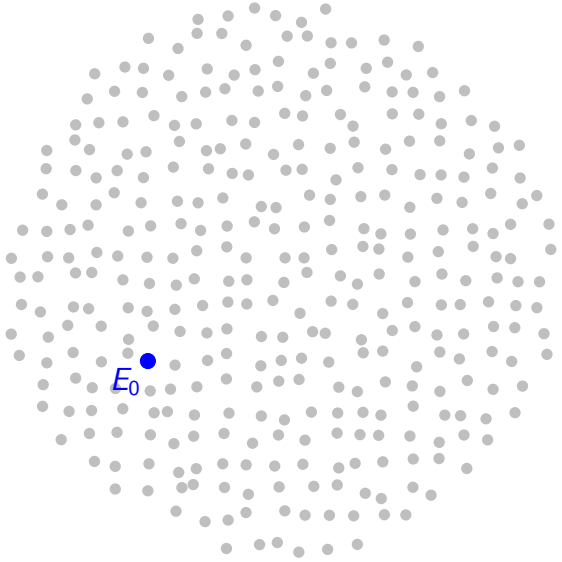

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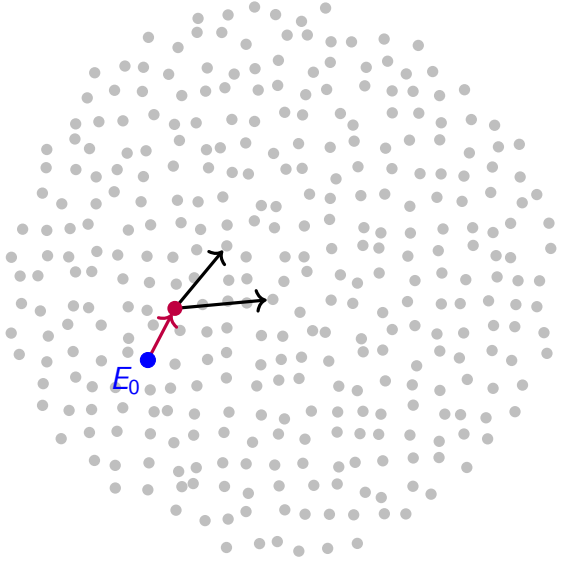

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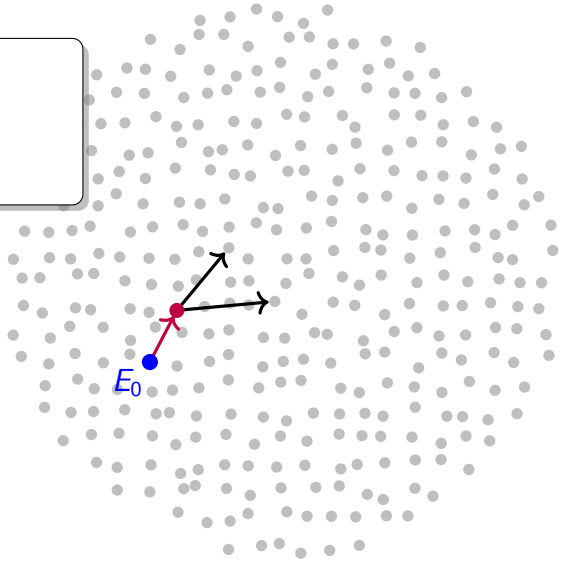

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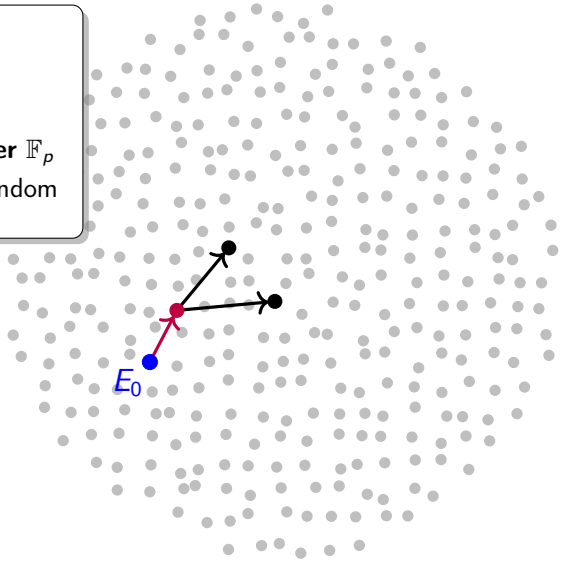

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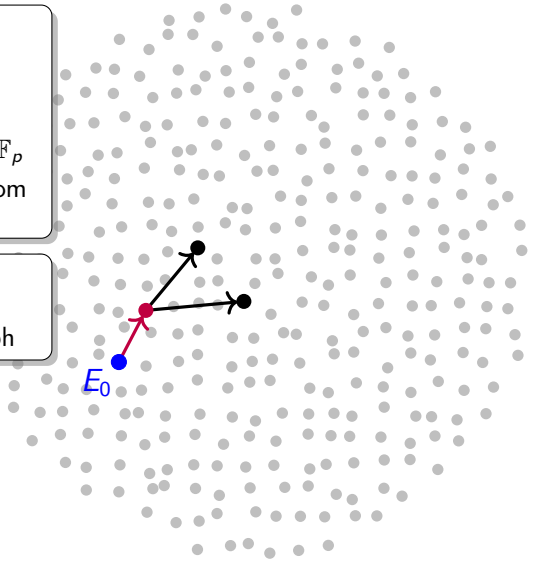

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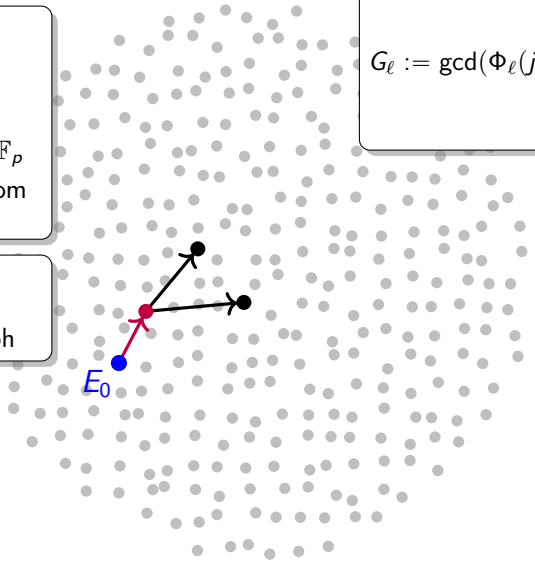

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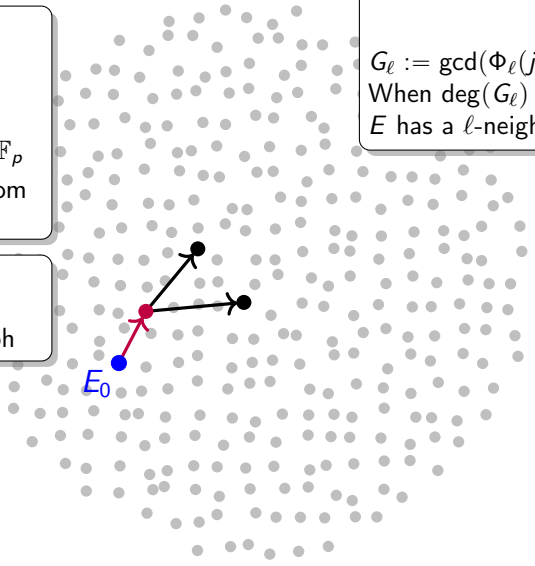

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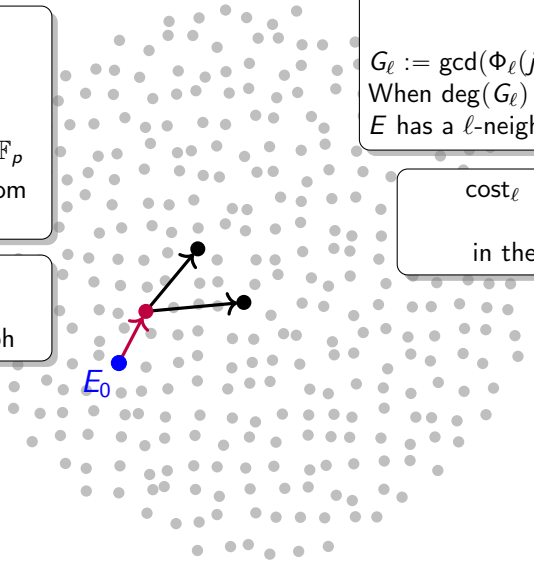

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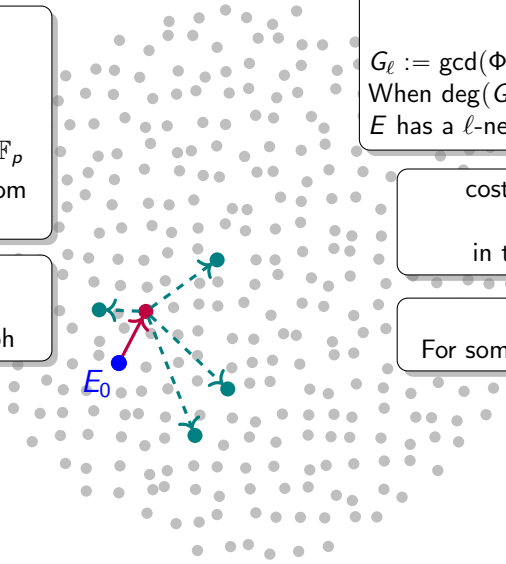

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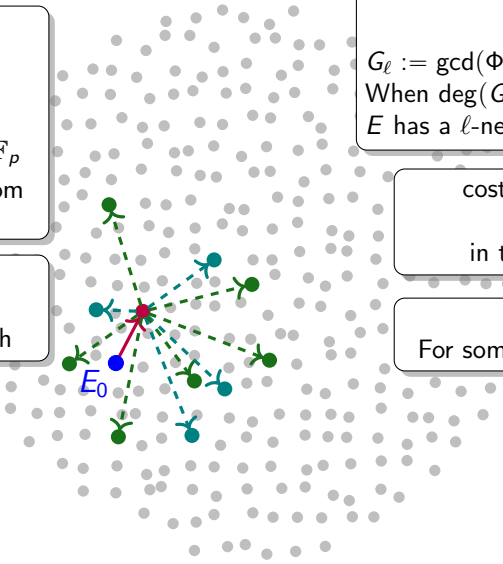

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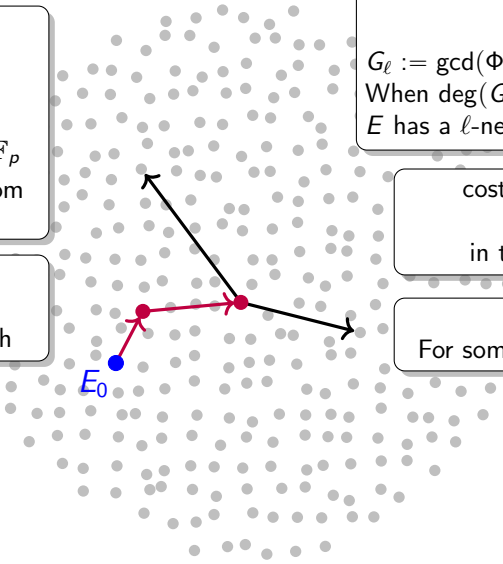

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SuperSolver's results [CCS22]

Core idea

Perform **fast subfield root detection** for all ℓ in the subset of $L := \{\ell \text{ prime such that } \text{cost}_2 > \text{cost}_\ell\}$ that minimizes the total cost of each step

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Result

The SuperSolver algorithm has the **same asymptotic complexity** as the Delfs-Galbraith algorithm but is more than 17 times **more efficient in practice** (for large p).

Starting point of this **ongoing** work

From now on, this is joint work with Maria Corte Real Santos, David Jao, Joseph Macula, Michael Meyer, Travis Morrison, and Eli Orvis.

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New detection tool: OrientationFinder

Theorem

For some primes ℓ_1, ℓ_2, there exists a precomputable polynomial $R_{\ell_1,\ell_2}(X, Y) \in \mathbb{Z}[X, Y]$ such that

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Thus, the explicit orientation is $\mathbb{Z}[\sqrt{-p\ell_2}] \hookrightarrow \mathrm{End}(E'), \sqrt{-p\ell_2} \mapsto \varphi \circ \pi$.

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Remarks: it also gives an oriented curve by $\mathbb{Z}[\sqrt{-p}]$ (Delf-Galbraith) or $\mathbb{Z}[\sqrt{-\ell_2 p}]$ ("New")

The latter corresponds to $E \xrightarrow[\ell_1\text{-isogenous}]{} E' \xrightarrow[\ell_2\text{-isogenous}]{} E'^{(p)}$,i.e. $\exists \varphi : E' \to E'^{(p)}$ s.t. $\deg \varphi = \ell_2$.

Thus, the explicit orientation is $\mathbb{Z}[\sqrt{-p\ell_2}] \hookrightarrow \mathrm{End}(E'), \sqrt{-p\ell_2} \mapsto \varphi \circ \pi$.

These orientations were already studied in [CS22], where the question of such Generalized Delfs–Galbraith algorithms is raised.

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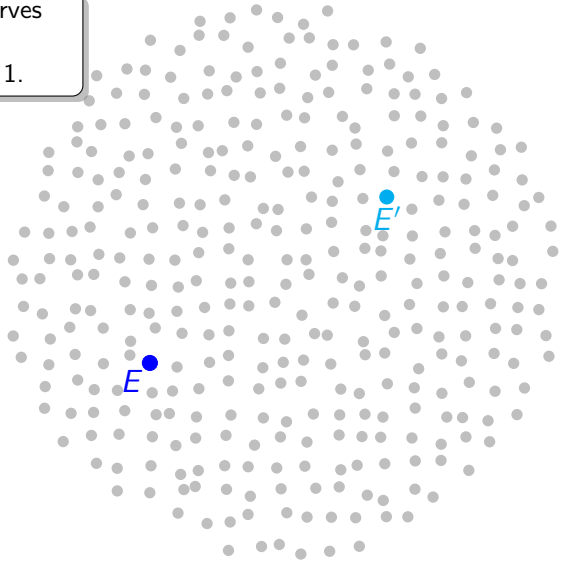

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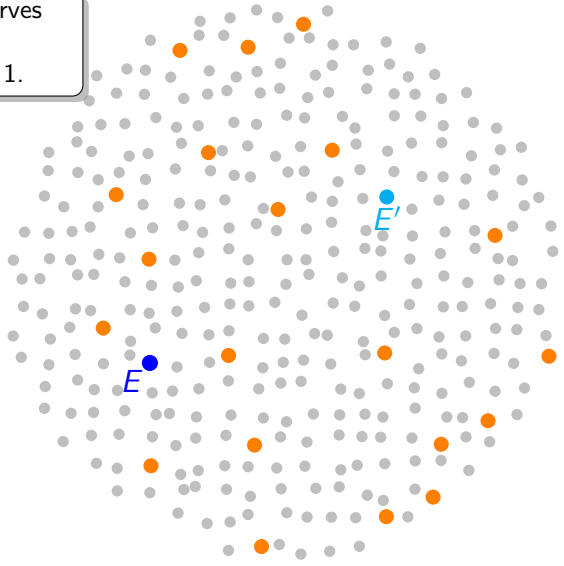

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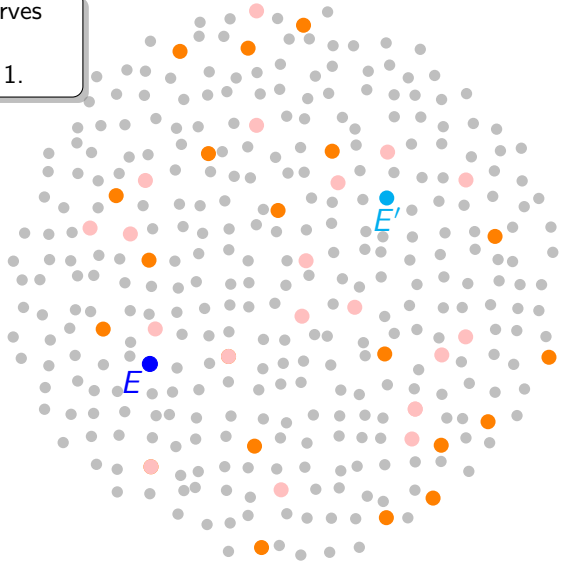

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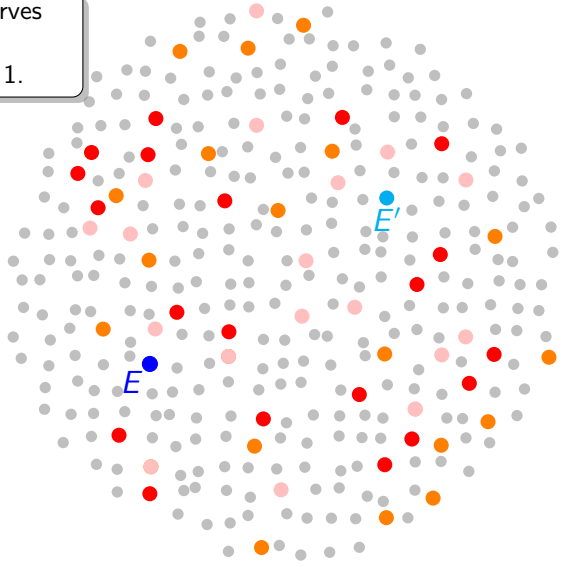

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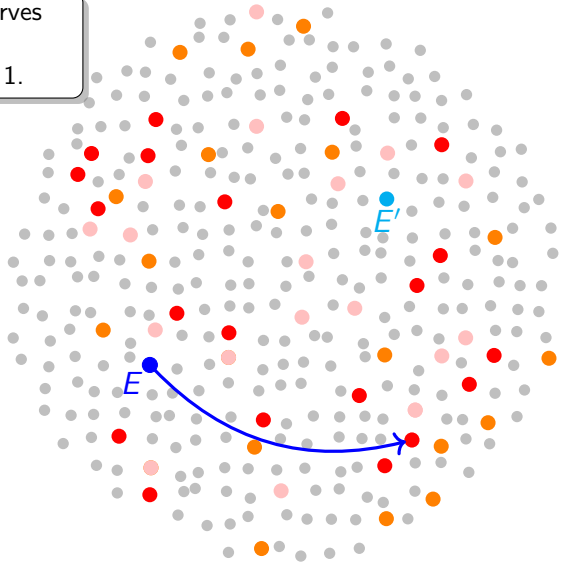

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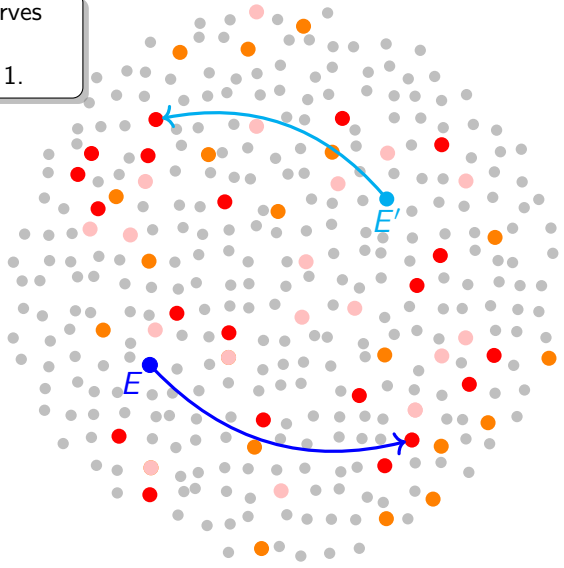

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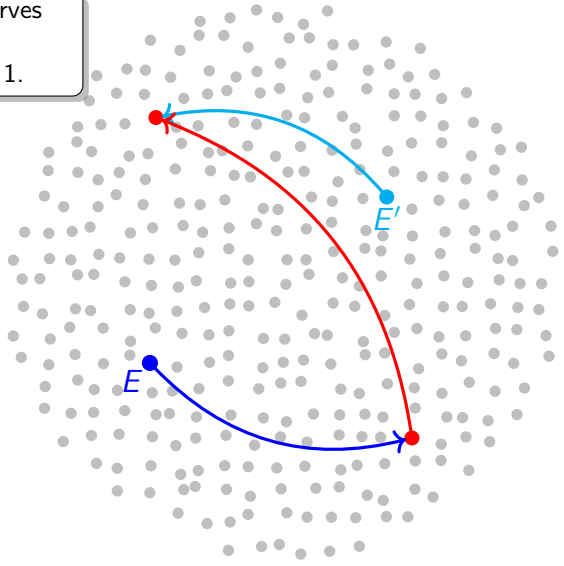

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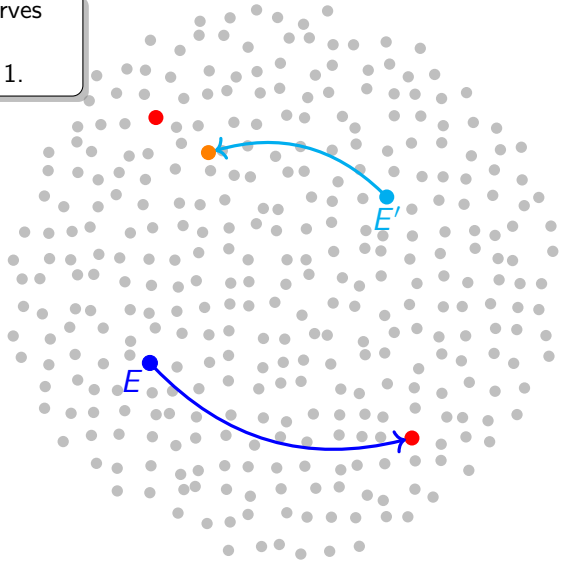

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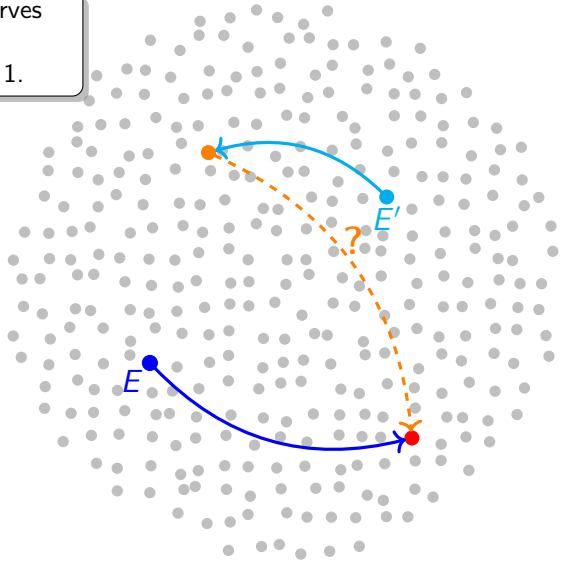

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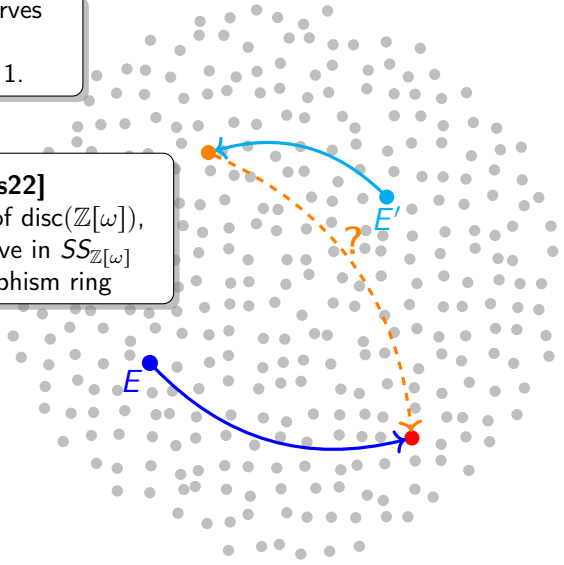

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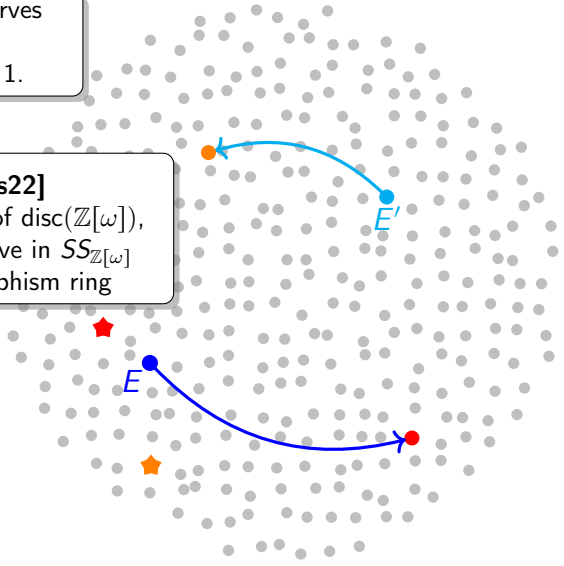

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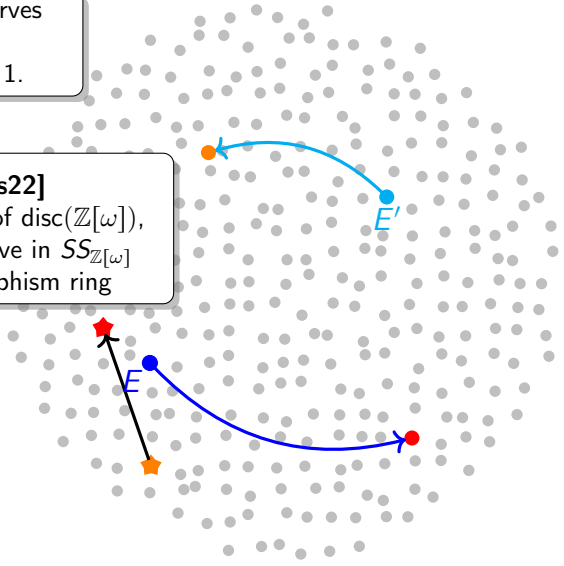

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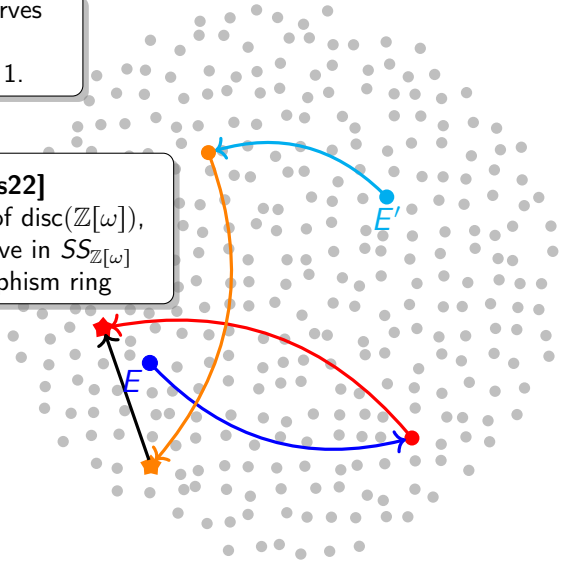

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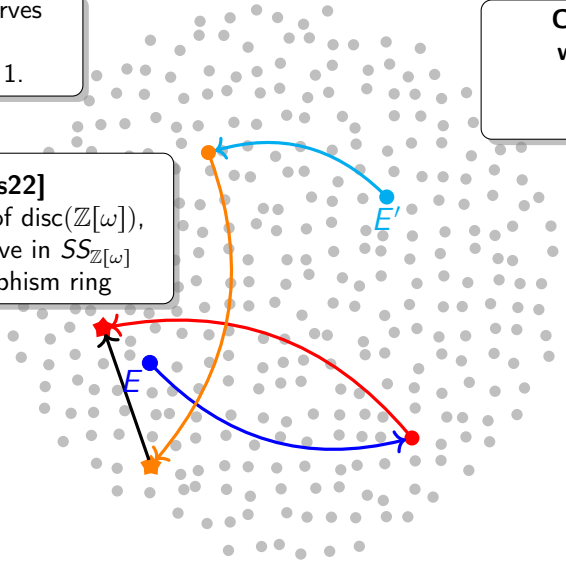

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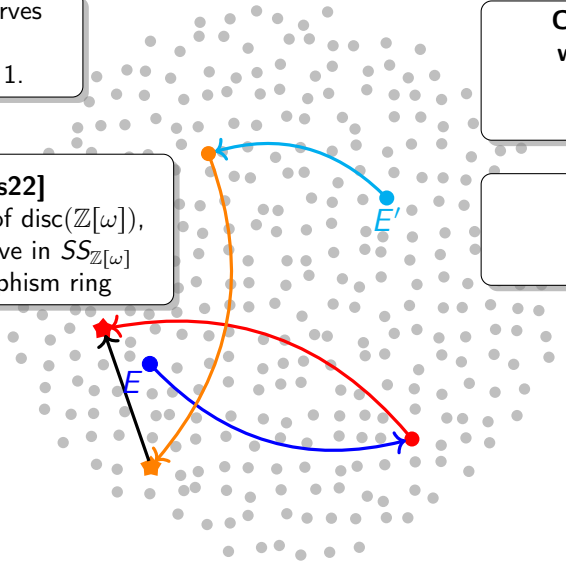

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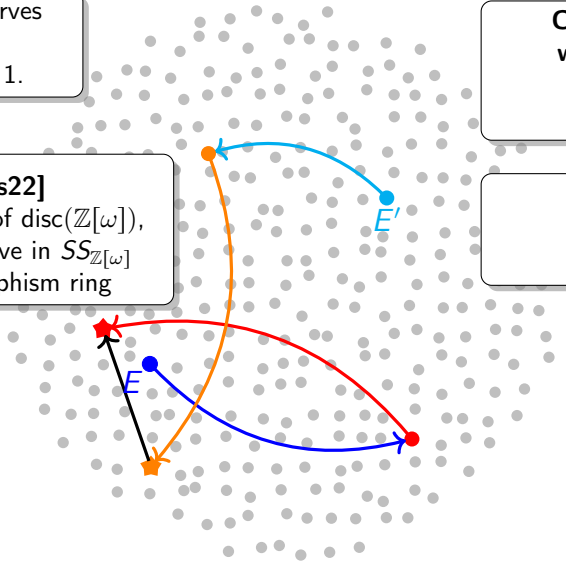

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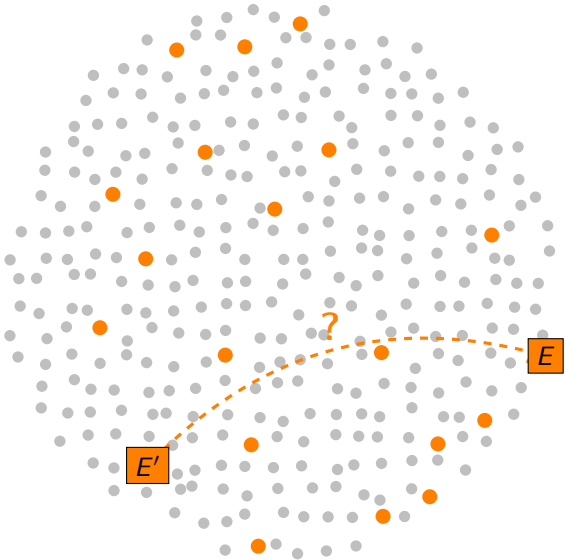
Let's adapt low-storage algorithms to this context [GHS02; GS13].



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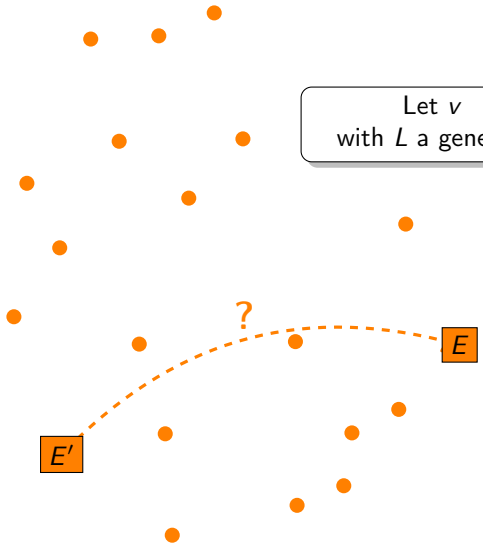

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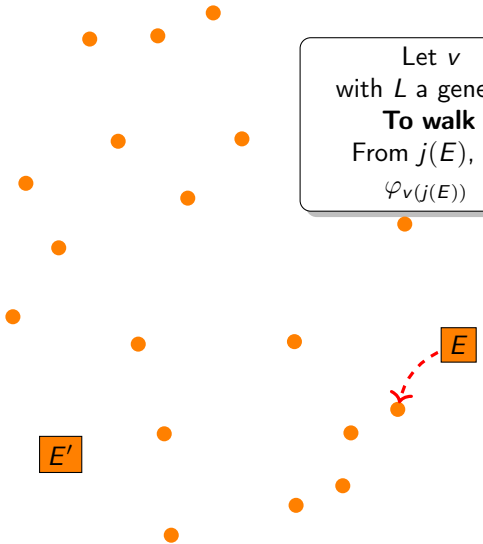

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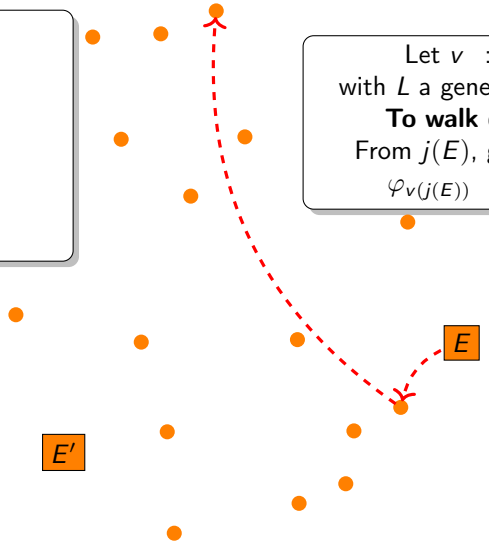

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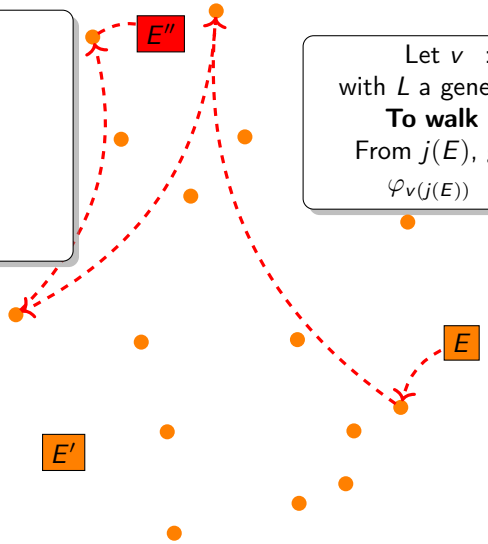

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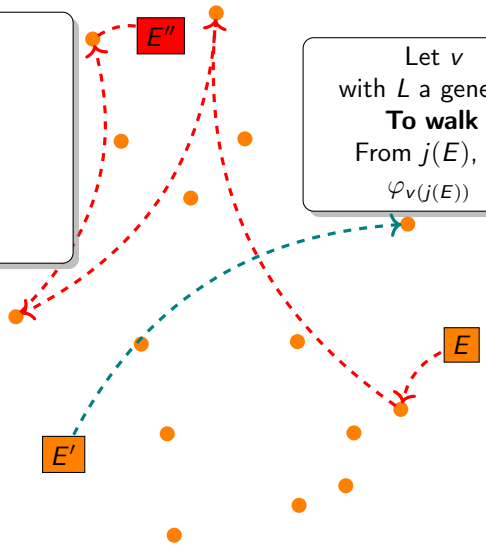

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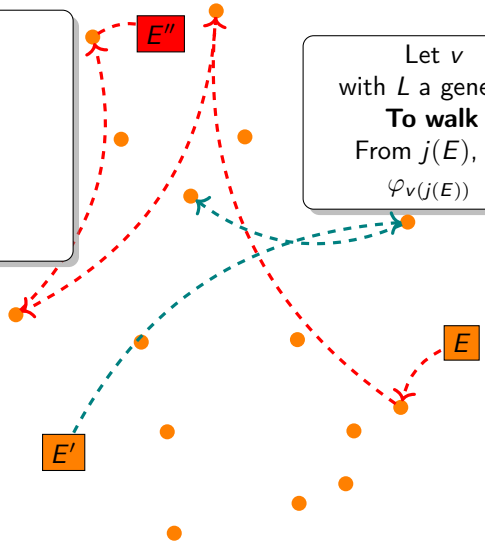

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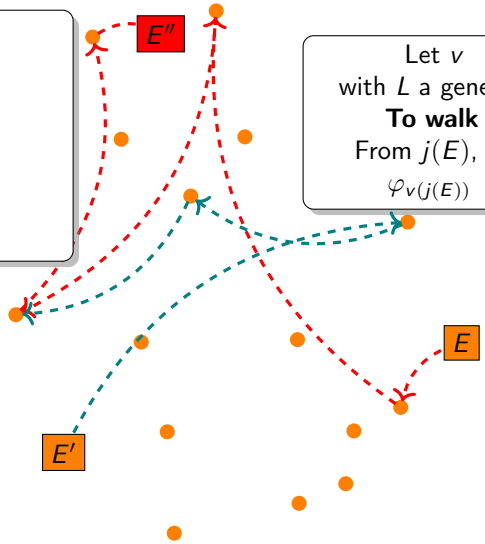

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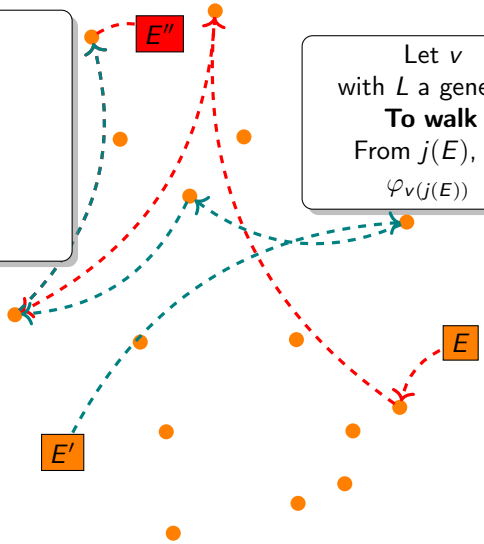

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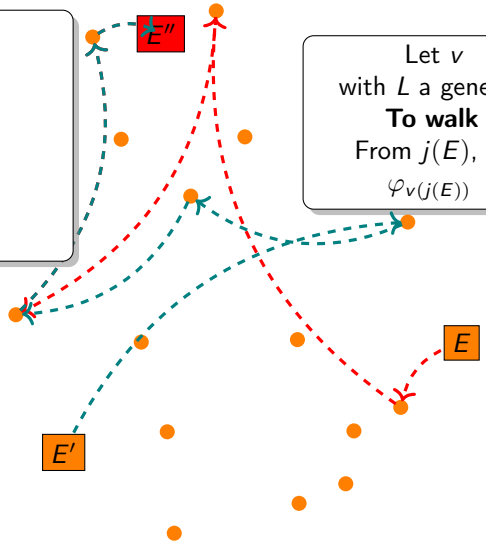

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What about the memory?

Let's adapt low-storage algorithms to this context [GHS02; GS13].

Method:

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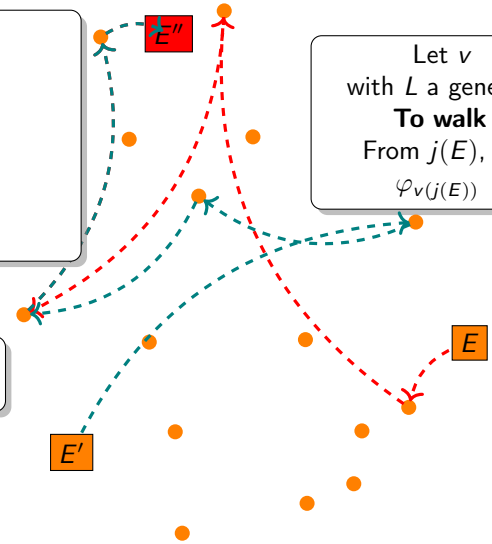

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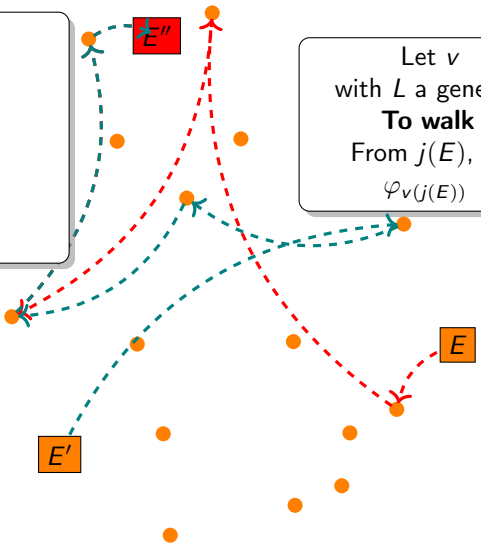

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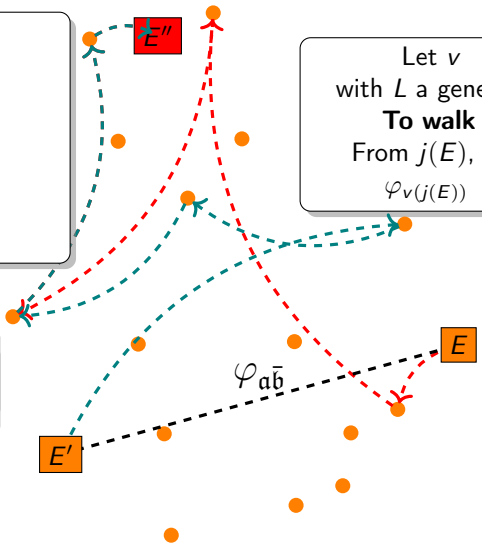

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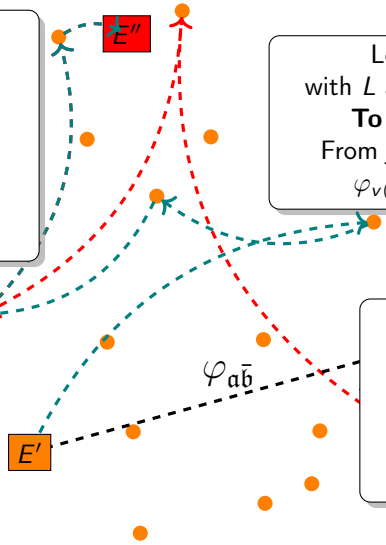

Complexity of this step:
Time: $\tilde{O}(p^{1/4})$
Memory: polylog(p)
Total complexity:
Time: $\tilde{O}(p^{1/2})$
Memory: polylog(p)

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Thank you for your attention!

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