Wagner's Algorithm Provably Runs in Subexponential Time for SIS$^\infty$

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Context

- Shor's algorithm
- Emergence of quantum computers
- NIST calls for **post-quantum cryptography** standardization
  - Kyber, Dilithium, Falcon (lattices)
  - Sphincs+ (hash functions)
  - HQC (codes)

Lattice-based cryptography

- Dilithium relies on the SIS^∞ problem

Short Integer Solution in infinity norm ($\mathrm{SIS}^\infty_{n,m,q,\beta}$)

Let be $n, m, q \in \mathbb{N}$ and $\beta > 0$. Given a uniformly random matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, find a **non-zero** vector $\mathbf{x} \in \mathbb{Z}^m$ such that

- $\mathbf{A}\mathbf{x} = \mathbf{0} \bmod q$
- $\|\mathbf{x}\|_\infty \leq \beta$

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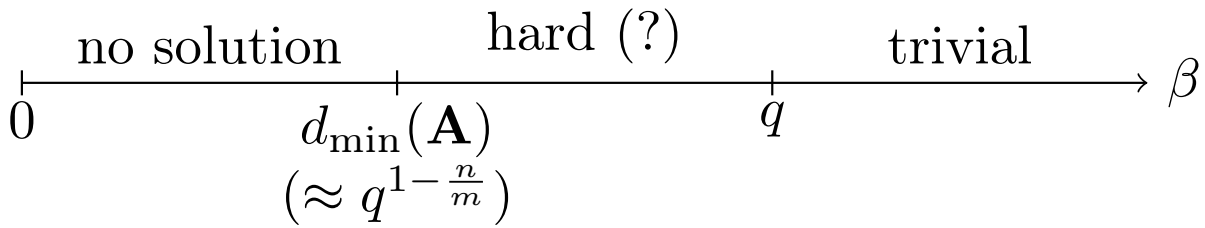

Lattice-based cryptography

Lattice

Given a basis $\mathbf{B} := (\mathbf{b}_1, \ldots, \mathbf{b}_k) \in \mathbb{R}^{d \times k}$, the *lattice* generated by $\mathbf{B}$ is the set of all integer linear combinations of the basis vectors $\mathbf{b}_i$, i.e.,

$$\mathcal{L}(\mathbf{B}) := \left\{ \sum_{i=1}^{k} z_i \mathbf{b}_i : z_i \in \mathbb{Z} \right\} \subseteq \mathbb{R}^d.$$

Lattice-based cryptography

$\mathrm{SIS}^{\infty}_{n,m,q,\beta}$ matrix $\mathbf{A}$

Given $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, find a **non-zero** vector $\mathbf{x} \in \mathbb{Z}^m$ such that

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$$\Longleftrightarrow$$

β-SVP^{∞} in the lattice $\Lambda_q^{\perp}(\mathbf{A})$

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For simplicity, consider $\mathbf{A} = [\mathbf{A}'|\mathbf{I}_n]$ with $\mathbf{A}' \in \mathbb{Z}_q^{n \times (m-n)}$

$$\text{The basis } \mathbf{B} := \begin{pmatrix} 0 & \mathbf{I}_{m-n} \\ q\mathbf{I}_n & -\mathbf{A}' \end{pmatrix} \text{ generates } \mathcal{L}(\mathbf{B}) = \mathbf{B}\mathbb{Z}^m = \Lambda_q^{\perp}(\mathbf{A}).$$

Motivation of this work

$$\textbf{BKW} = \textbf{Wagner} + \textbf{Dual distinguishing}$$

(LWE) (SIS)

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Historic

- Algorithms to solve LWE are variants of [BKW03]
- [KF15] claimed to solve LWE with ternary secret in subexponential time
- [HKM18] found an issue in their proof for certain regimes ($m = \Theta(n)$)

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Questions

- Is there a provable variant of Wagner to solve SIS^∞ in subexponential time?
- Can we fix [KF15] for LWE?
- Does it threaten Dilithium?

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Questions

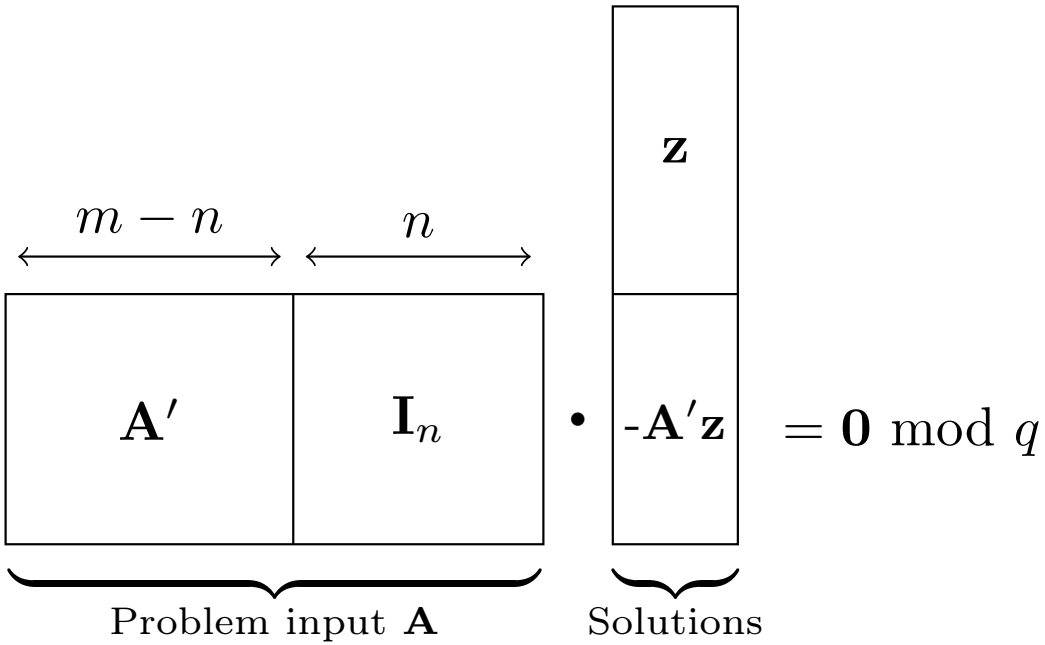
• Is there a provable variant of Wagner to solve SIS^∞ in subexponential time?	Yes for $\beta = \frac{q}{\mathrm{polylog}(n)}$
• Can we fix [KF15] for LWE?	Maybe?
• Does it threaten Dilithium?	No

Outline

1. Wagner-style algorithms to solve SIS^∞
2. A provable algorithm for SIS^∞
3. Implications for cryptographic problems

Outline

Main idea


$m-n$
n
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Problem input $\mathbf{A}$
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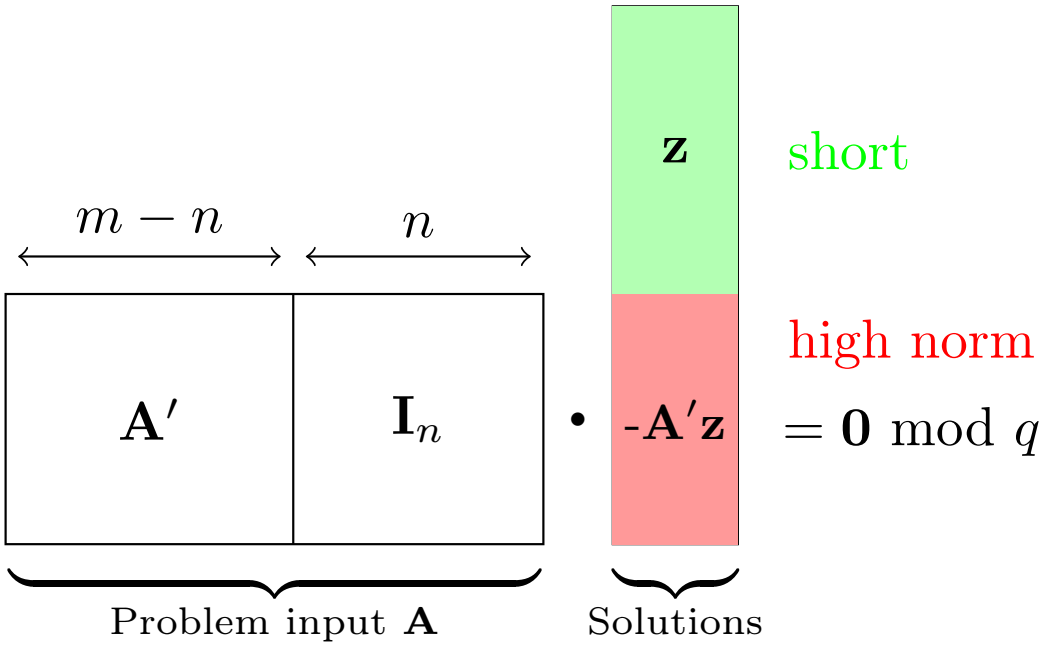
z
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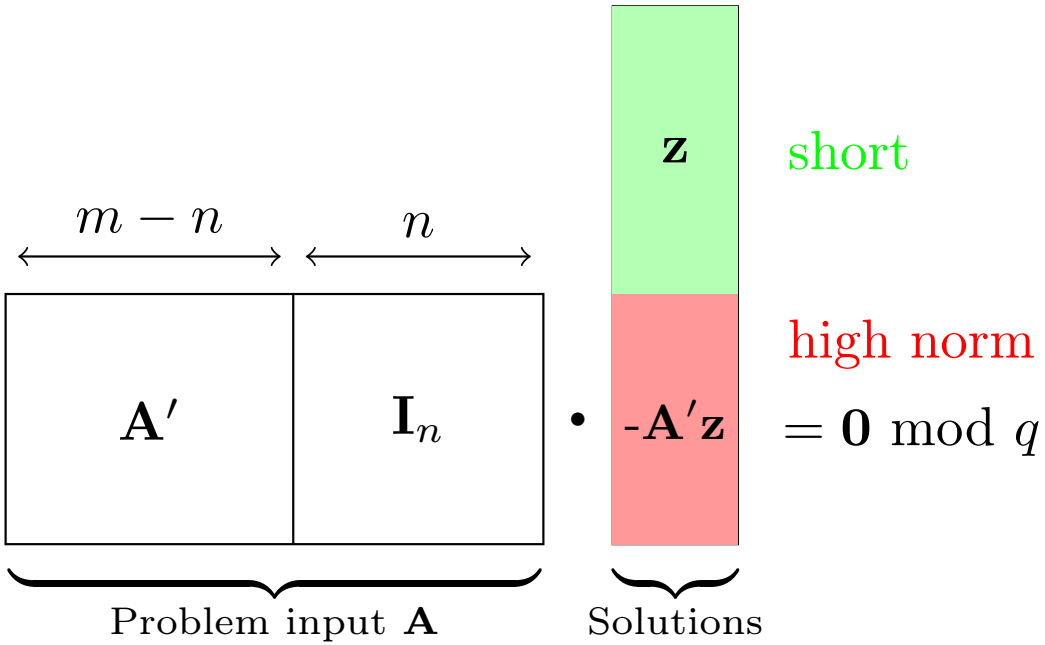
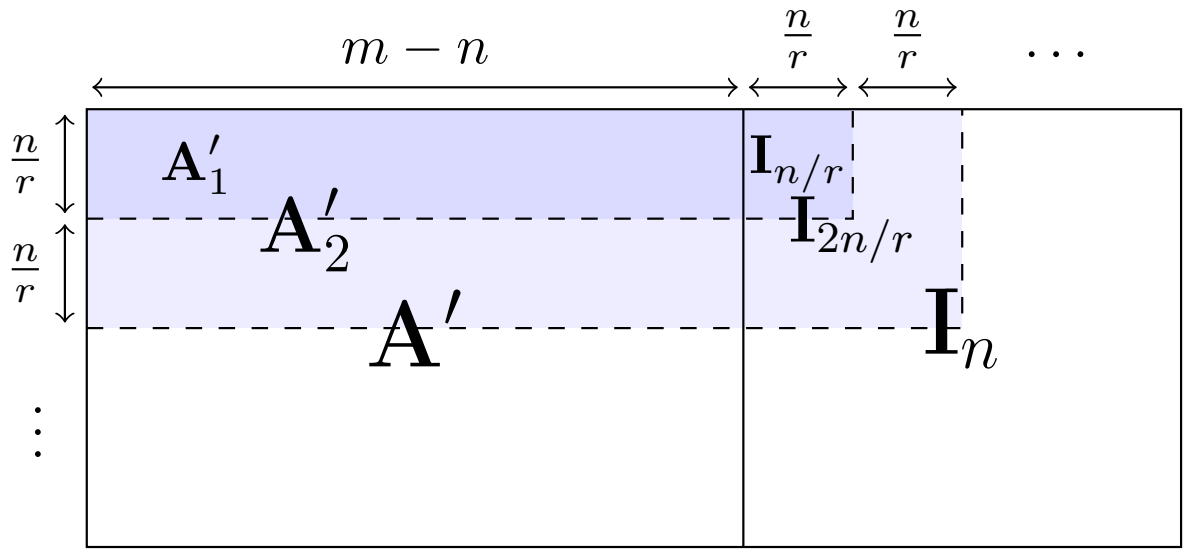


Wagner's algorithm [Wag02]

Input: list L
Output: tuples of elements in L that sum up to $\mathbf{0}$

Wagner's algorithm for SIS



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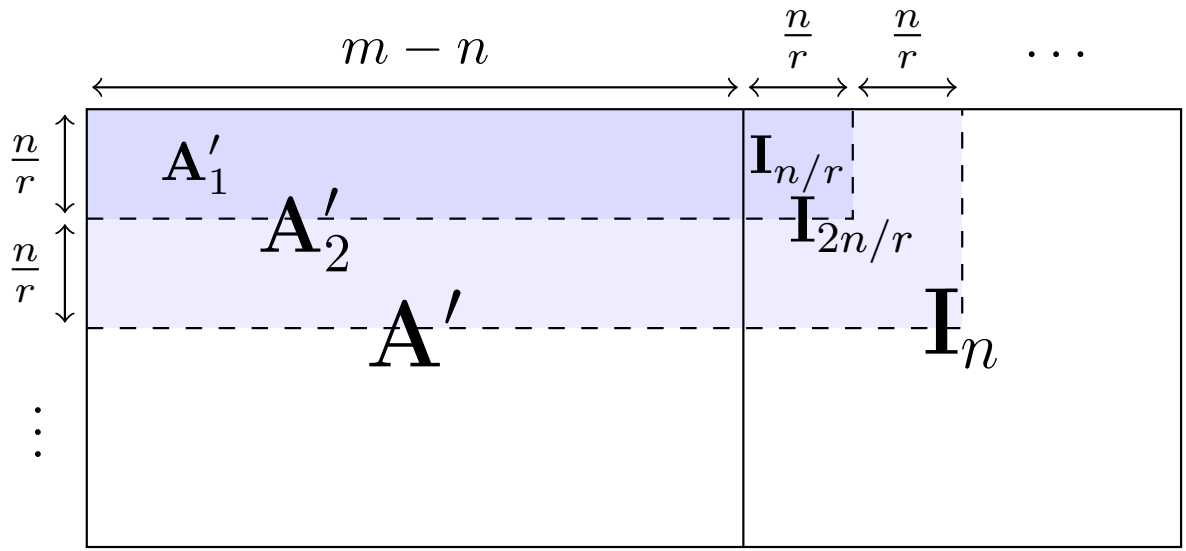


Wagner step in [BKW03]

Input: $\mathbf{A} = [\mathbf{A}' \mid \mathbf{I}_n] \in \mathbb{Z}_q^{n \times m}$, $\beta > 0$
Output: List of vectors $\mathbf{x} \in \mathbb{Z}_q^m$ such that $\mathbf{Ax} = \mathbf{0} \bmod q$ and $\|\mathbf{x}\|_\infty \leq 2^r \leq \beta$

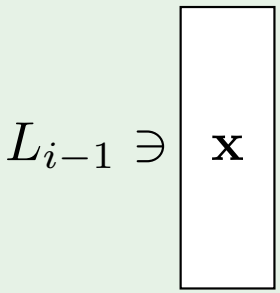
Divide $\mathbf{A}$ into submatrices $\mathbf{A}_i = [\mathbf{A}'_i|\mathbf{I}]$
Initialize a list L_0 with vectors from $\mathcal{U}(\{-1, 0, 1\}^{m-n})$
for $i = 1, \ldots, r$ **do**
    $L_i := \textbf{LiftAndCombine}(L_{i-1}, \mathbf{A}_i)$ $\quad \triangleright\ \forall \mathbf{x} \in L_i, \mathbf{A}_i\mathbf{x} = \mathbf{0} \bmod q$
return L_r

Wagner's algorithm for SIS

LiftAndCombine

In: List L_{i-1} of vectors $\mathbf{x} \in \mathbb{Z}_q^{(i-1)n/r}$ s.t. $\mathbf{A}_{i-1}\mathbf{x} = \mathbf{0} \bmod q$ and $\|\mathbf{x}\|_\infty \leq 2^{i-1}$

Out: List L_i of vectors $\mathbf{x} \in \mathbb{Z}_q^{in/r}$ such that $\mathbf{A}_i\mathbf{x} = \mathbf{0} \bmod q$ and $\|\mathbf{x}\|_\infty \leq 2^i$



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Lift: Compute $\mathbf{y} \in \mathbb{Z}_q^{n/r}$ such that

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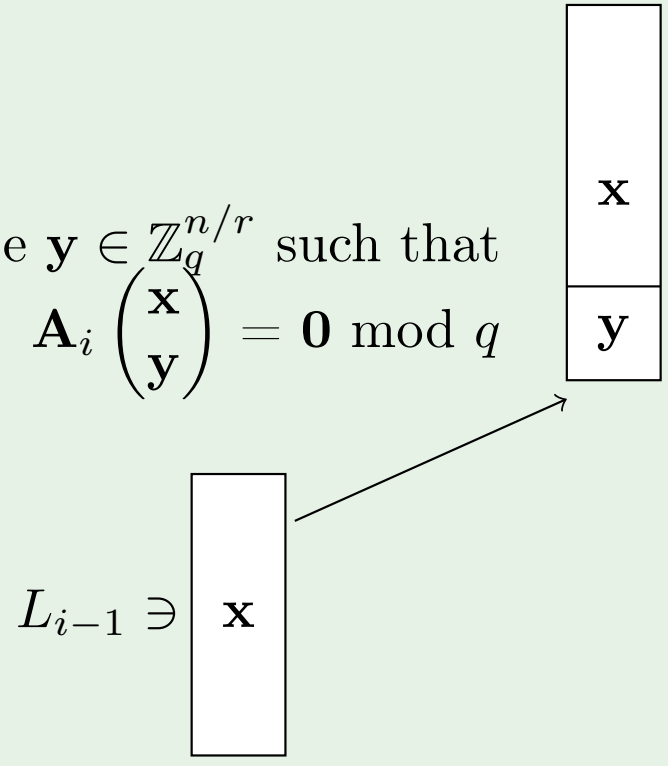

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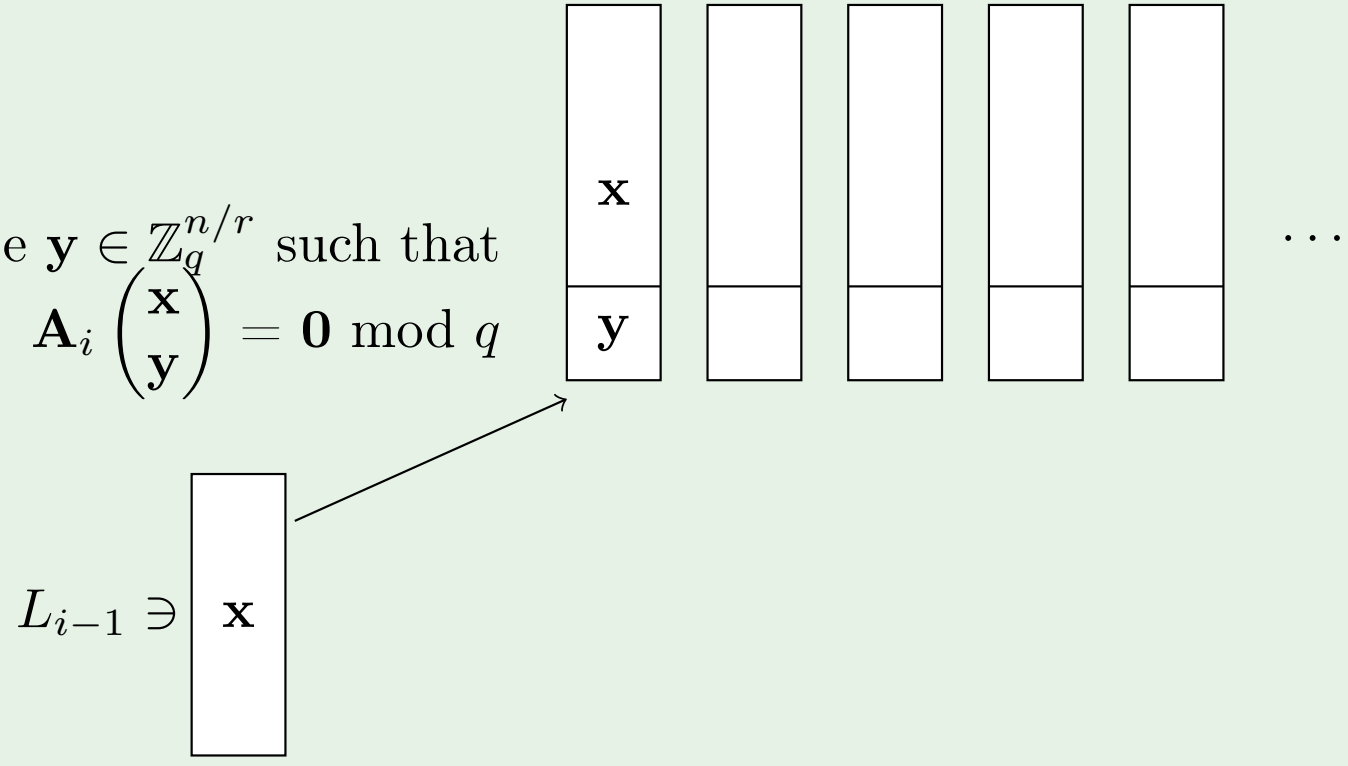

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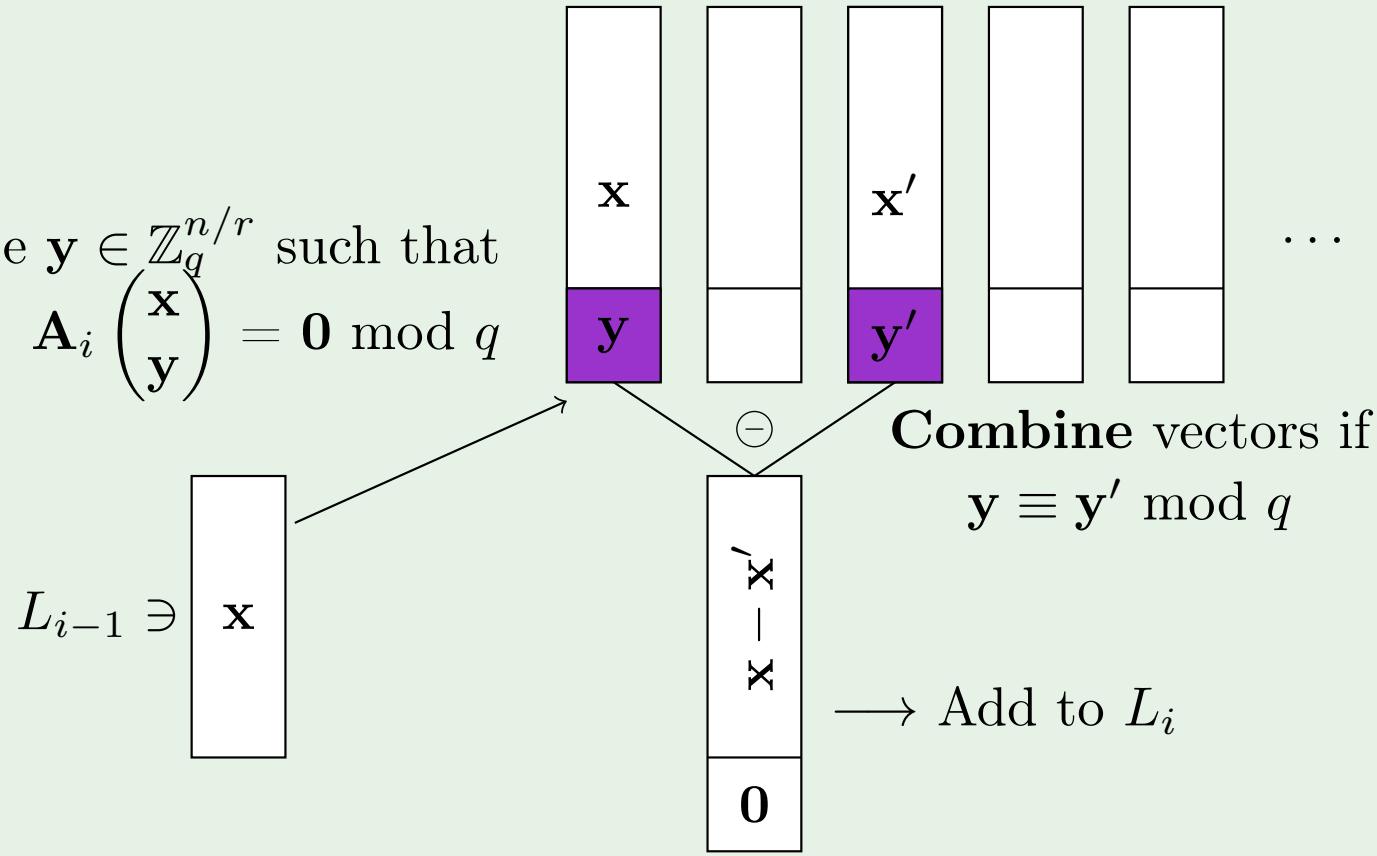

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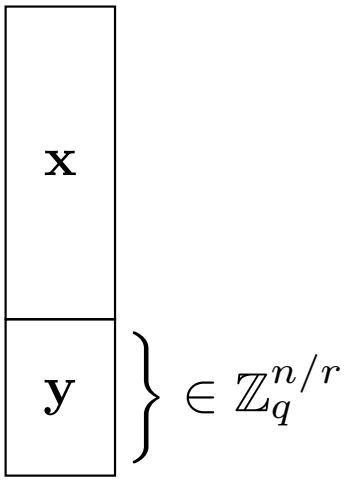

Time complexity: $O(r \cdot q^{n/r})$

Lazy-modulus switching

LiftAndCombine with lazy-mod switching [AFFP14]

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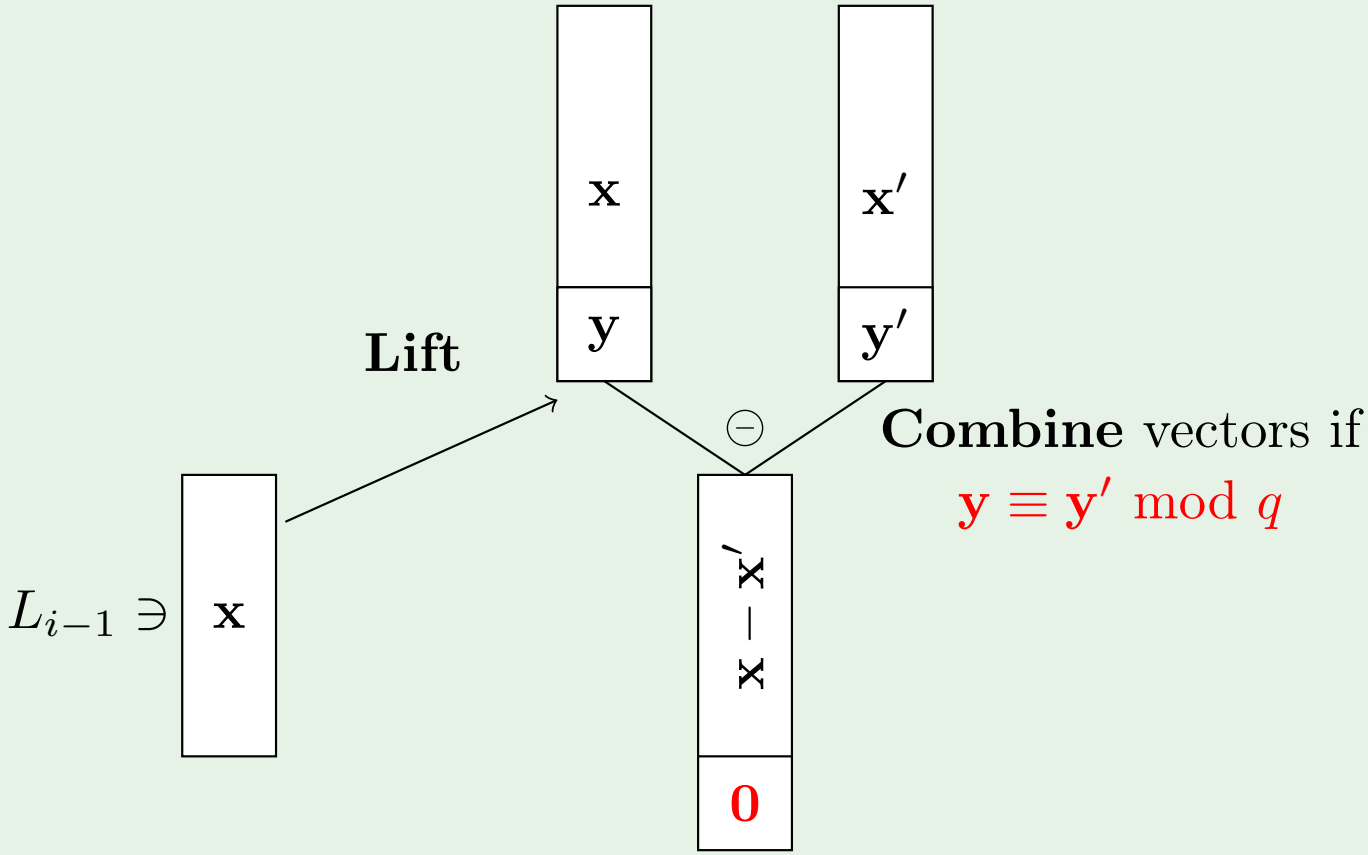

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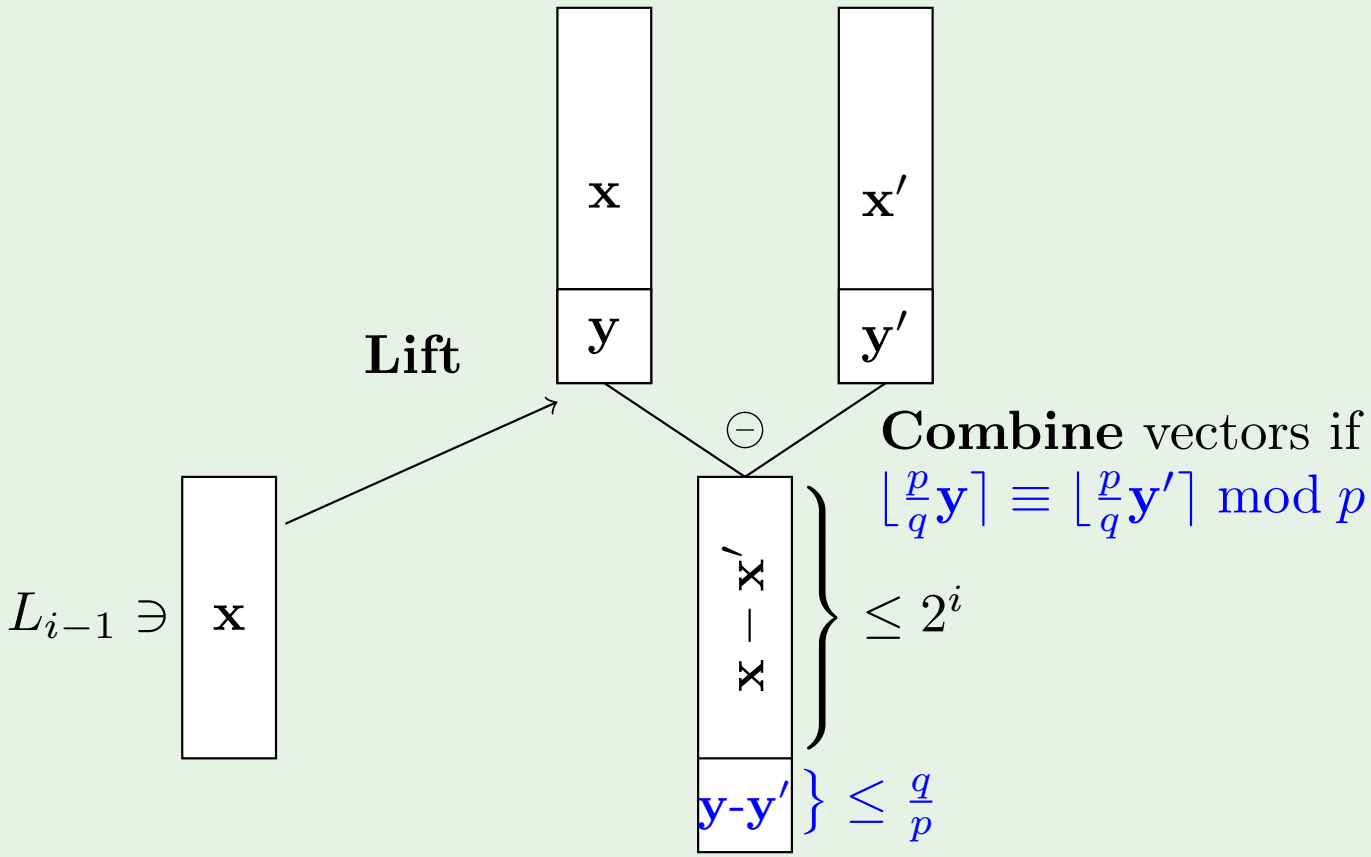

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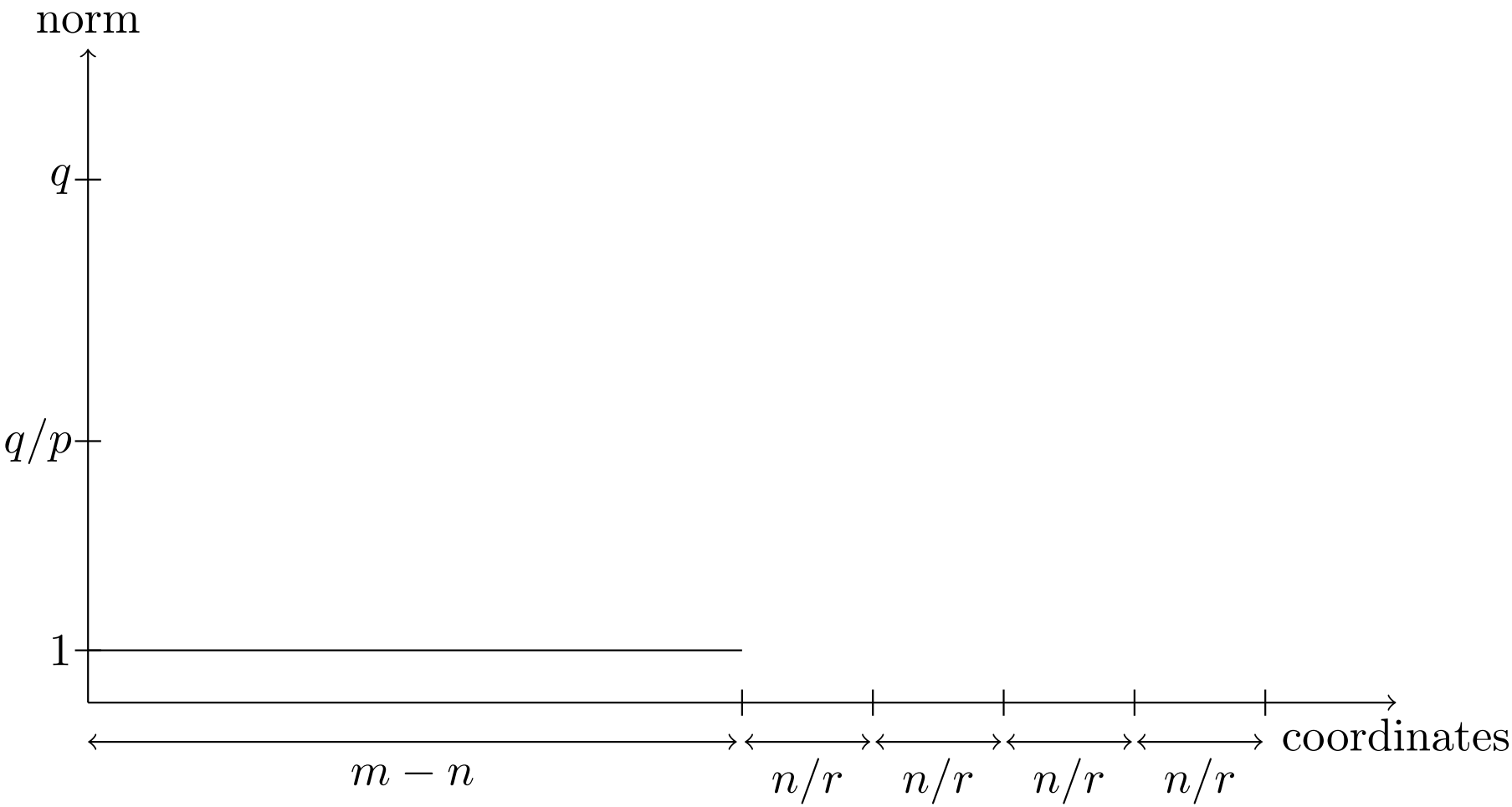
Bounds on the components of $\mathbf{x} \in L_0$ (Algorithm [AFFP14])



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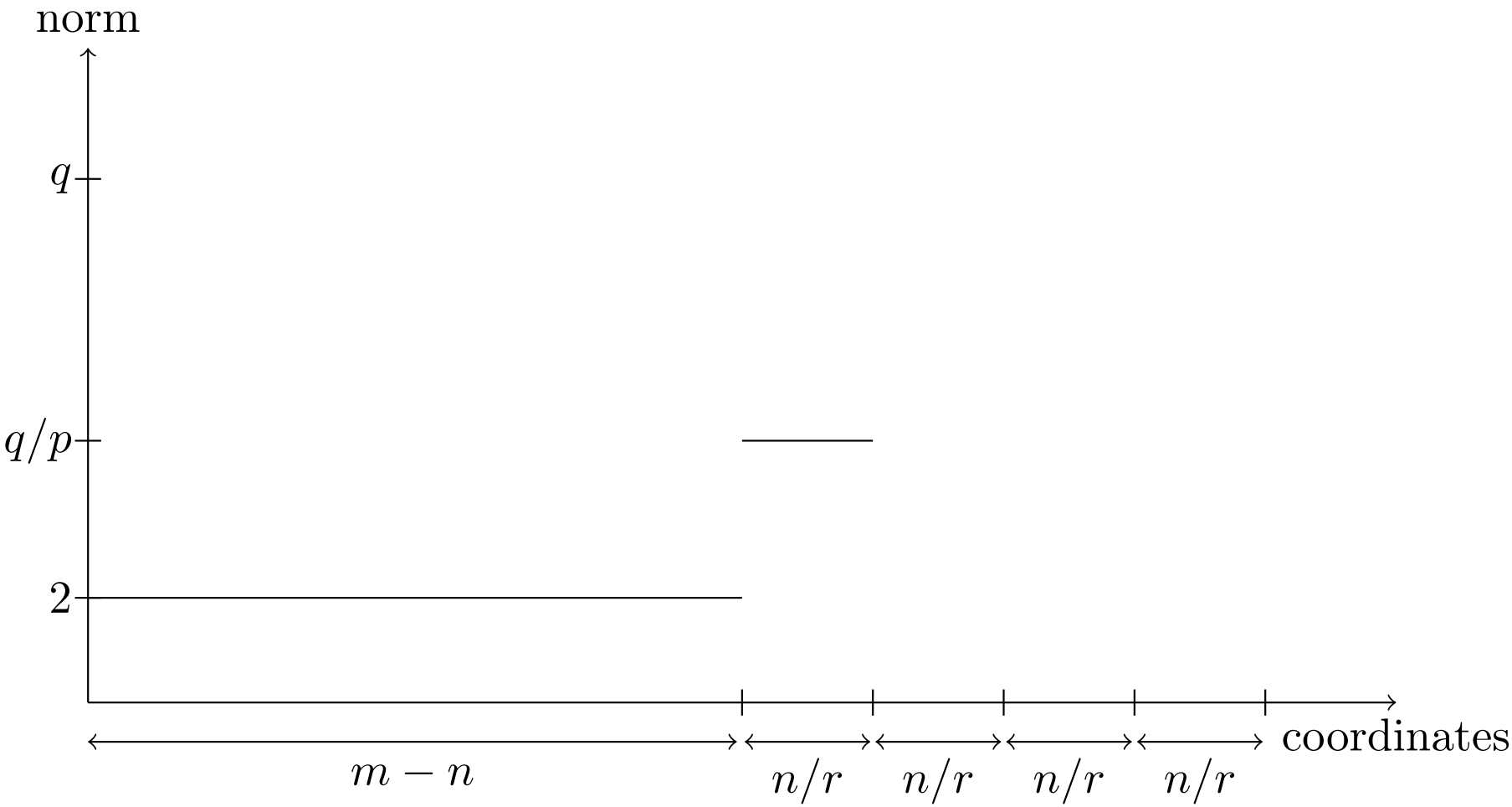
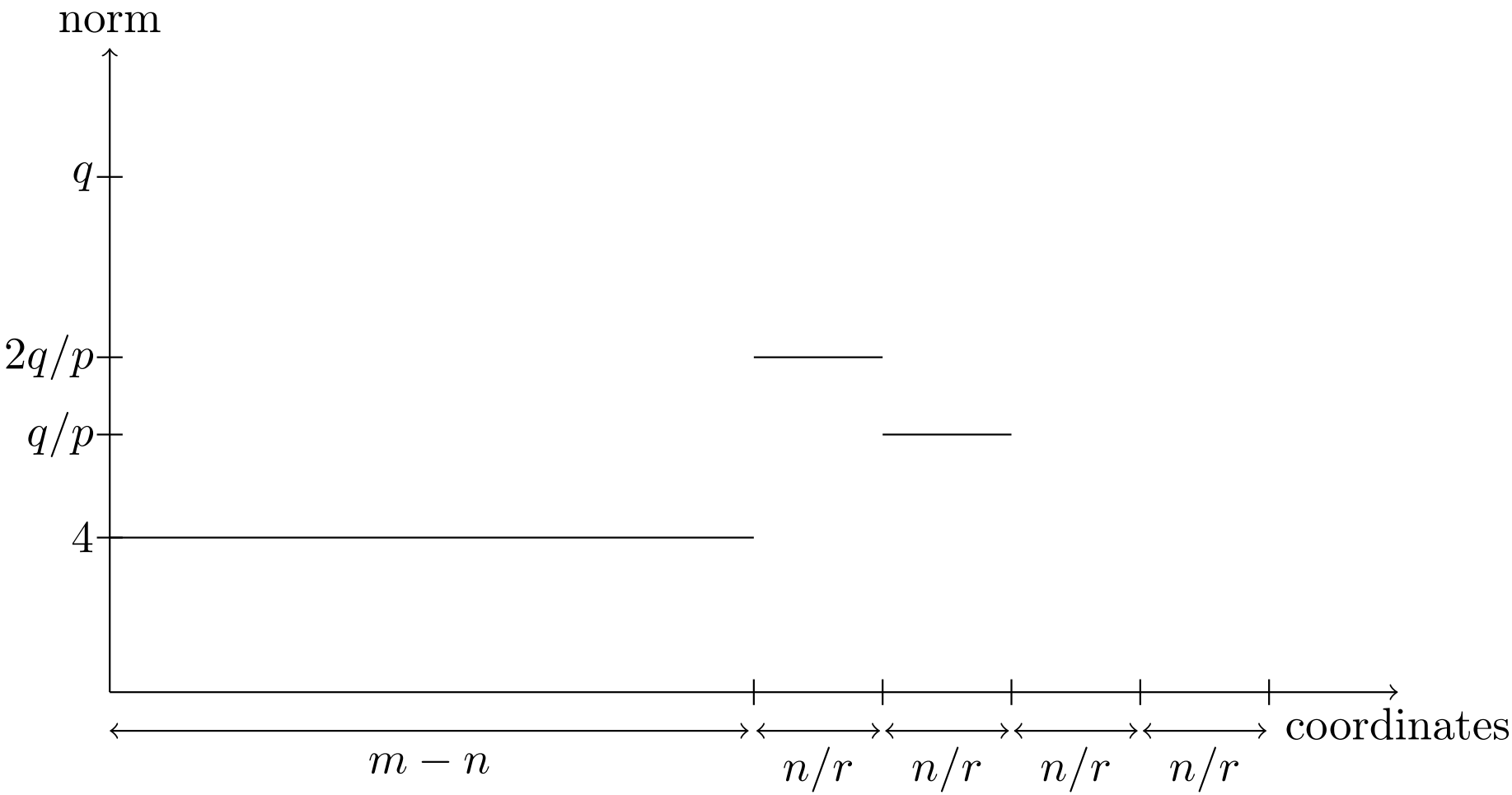
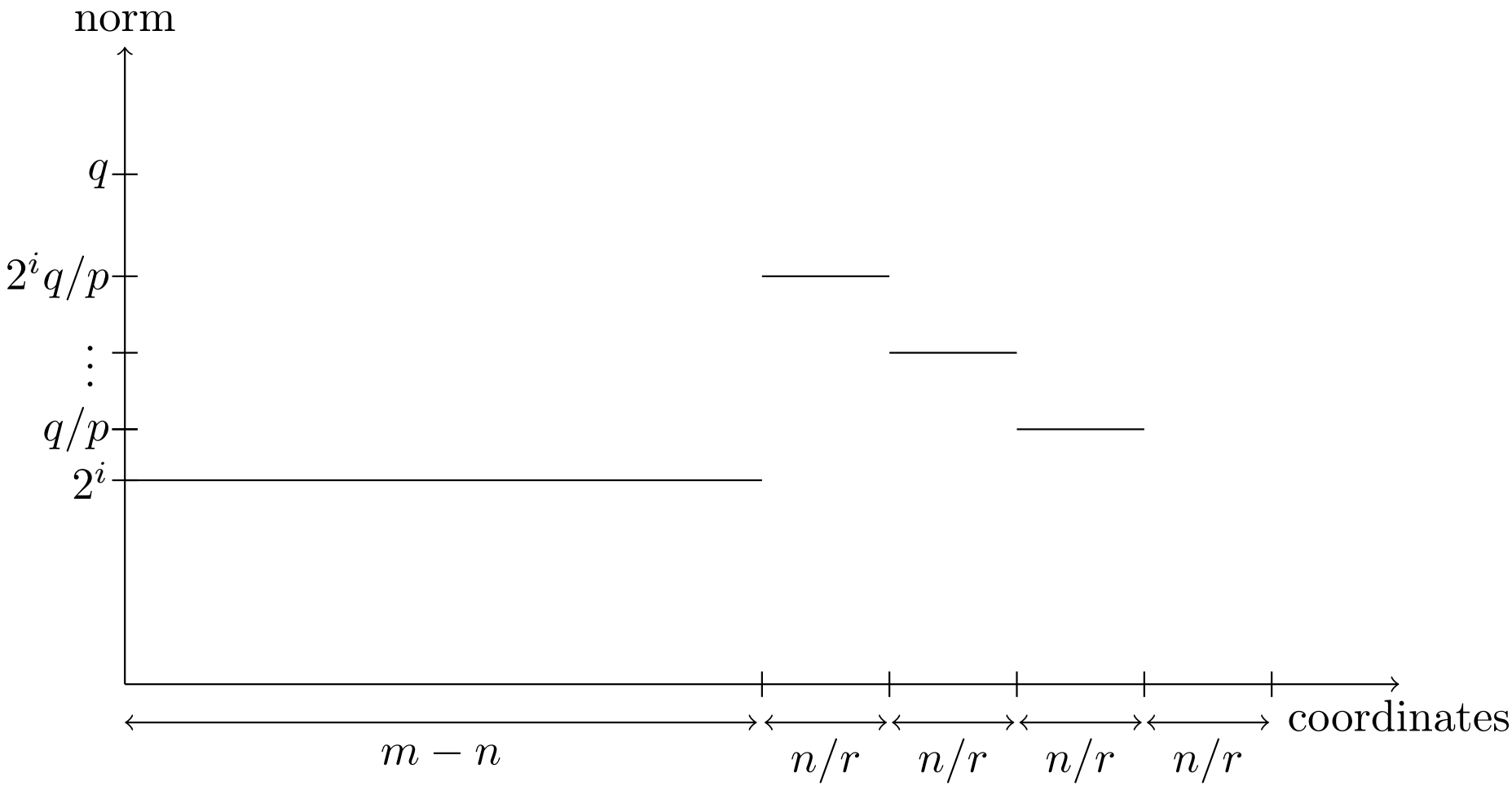


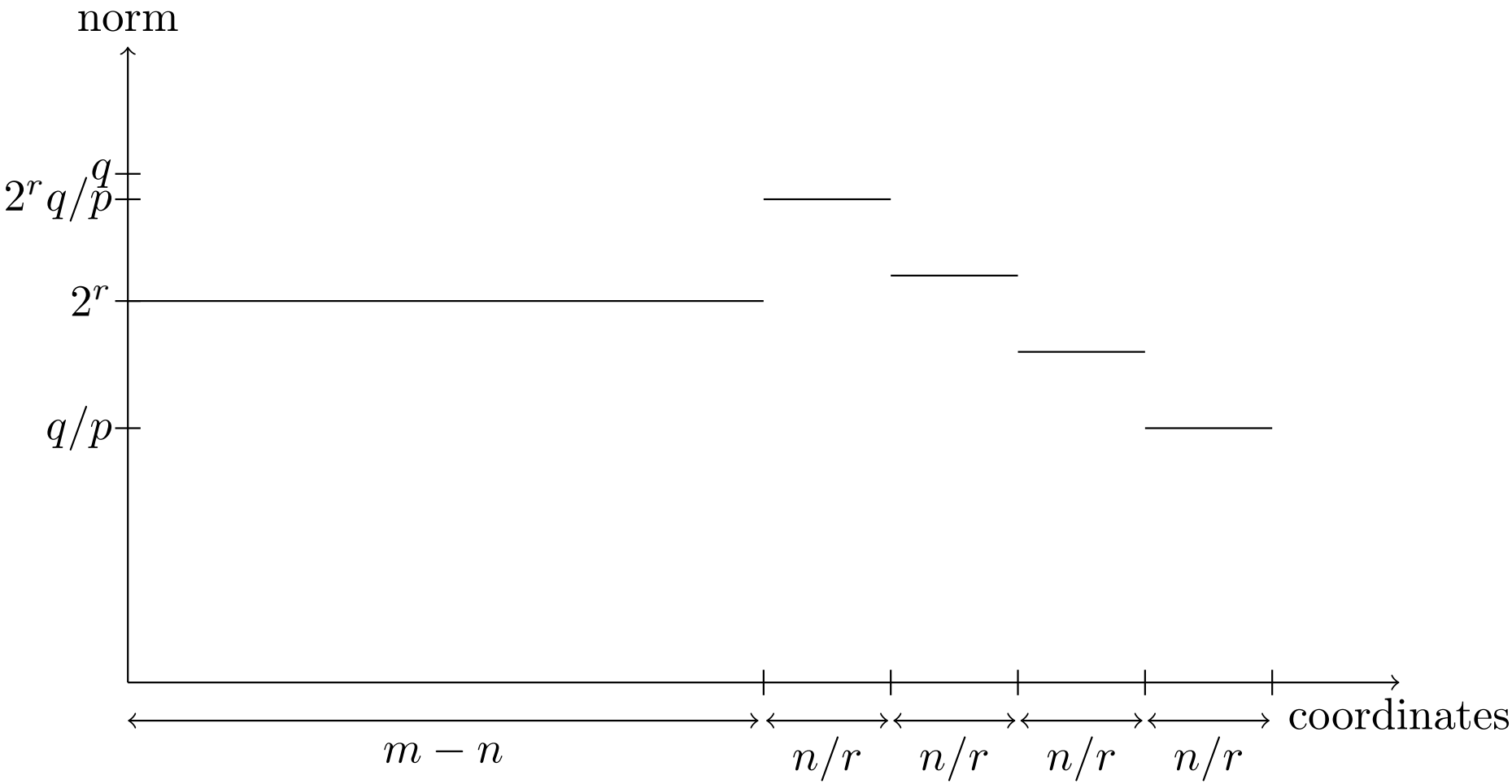
Bounds on the components of $\mathbf{x} \in L_2$ (Algorithm [AFFP14])



Bounds on the components of $\mathbf{x} \in L_i$ (Algorithm [AFFP14])

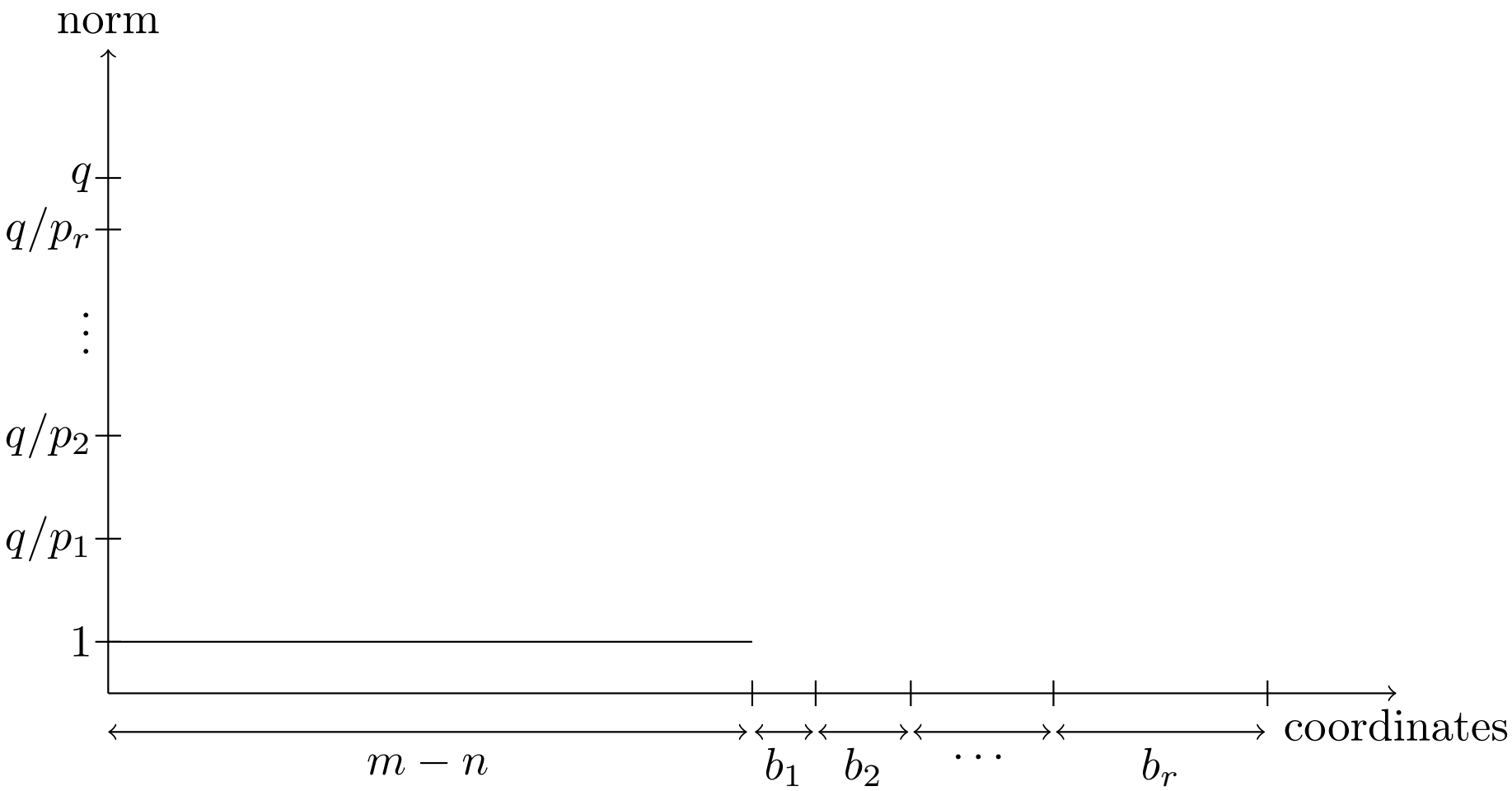


Bounds on the components of $\mathbf{x} \in L_r$ (Algorithm [AFFP14])



Kirchner-Fouque algorithm

Bounds on the components of $\mathbf{x} \in L_0$ (Algorithm [KF15])



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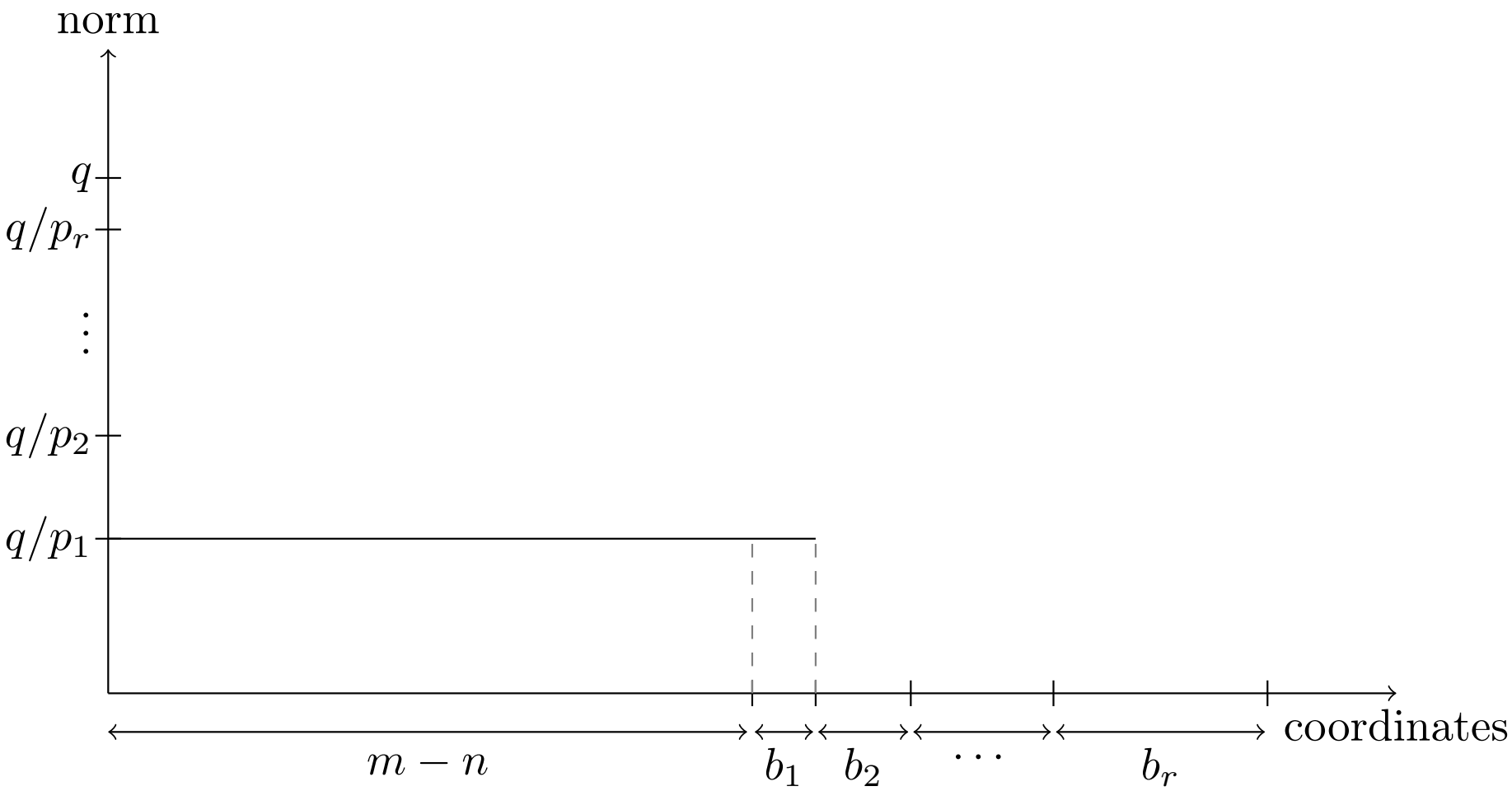

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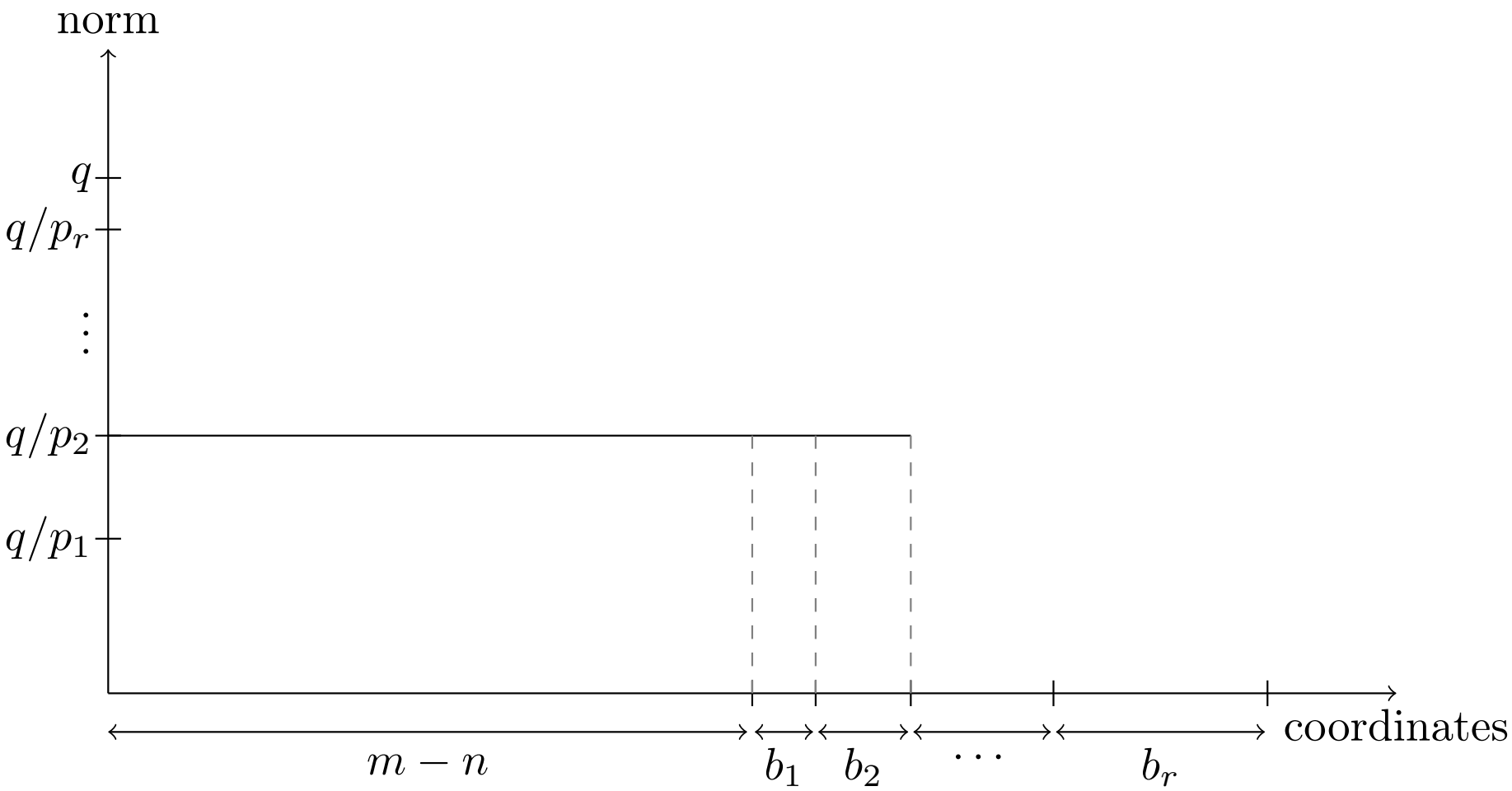

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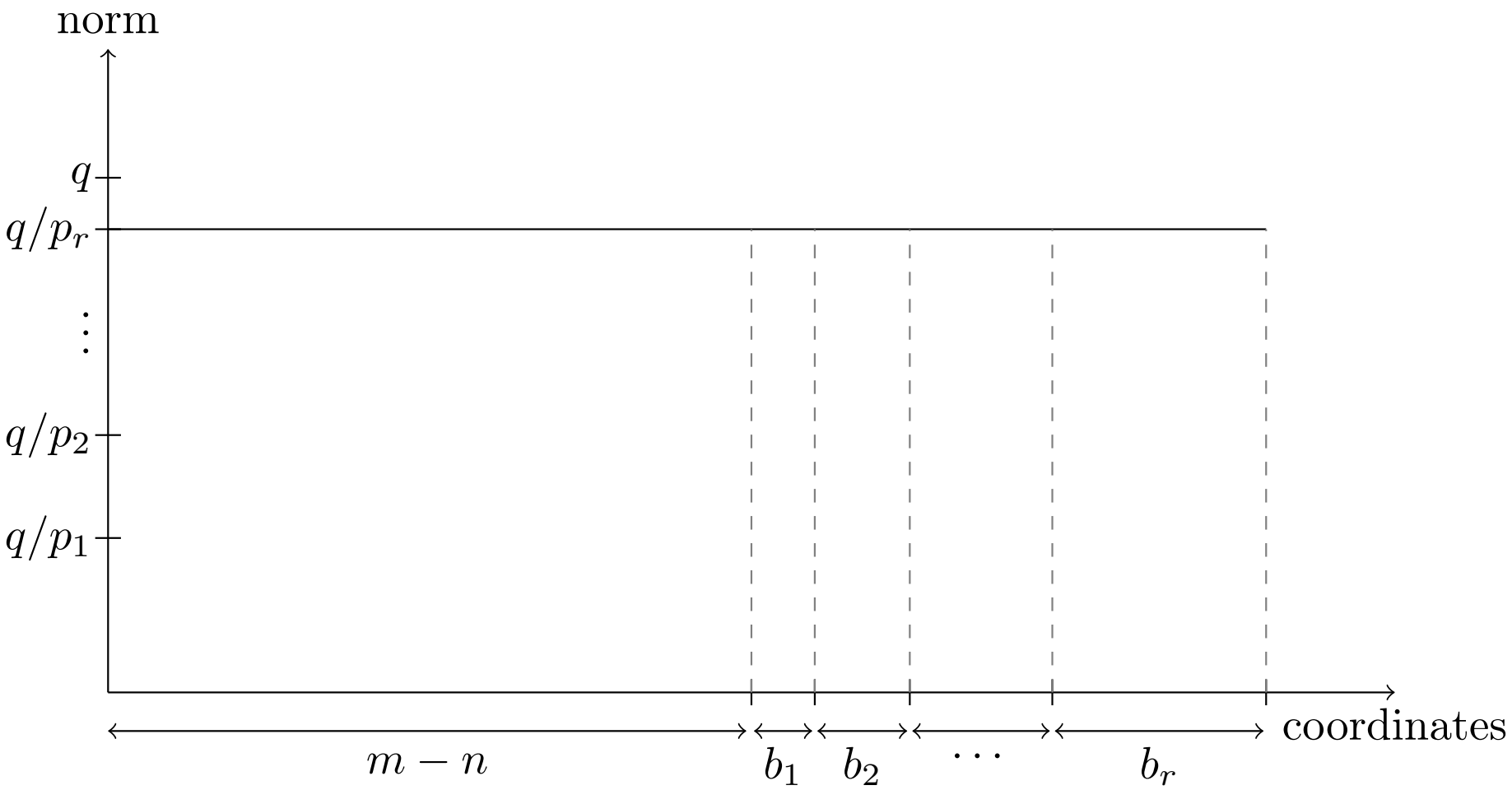
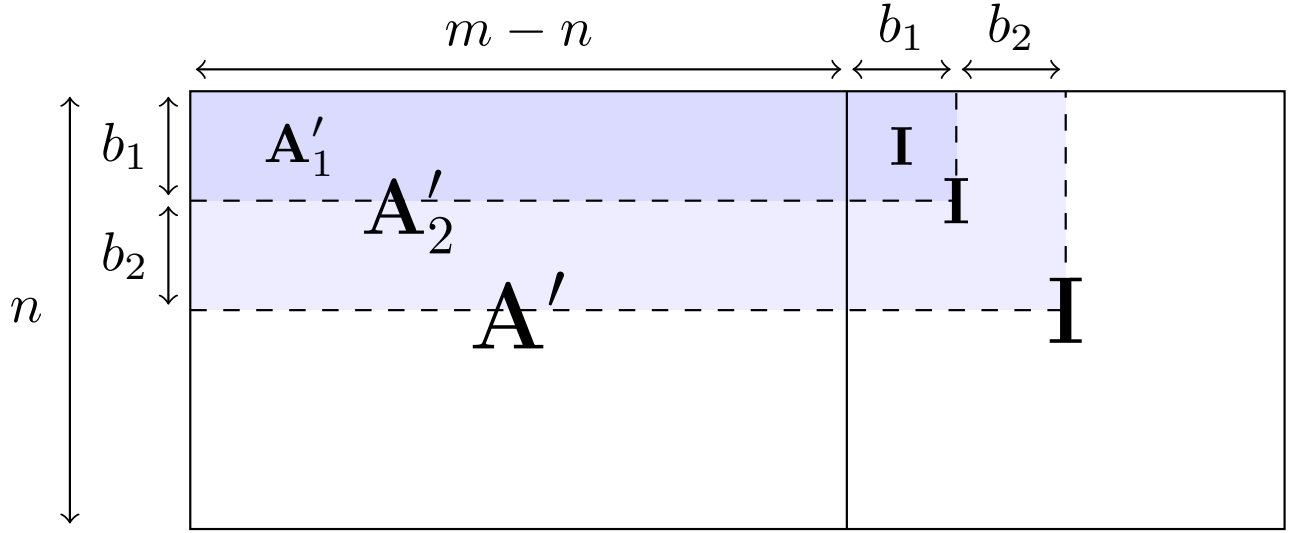


Kirchner-Fouque algorithm

Time complexity: $O(r \cdot |L_i|)$



Parameter selection for target norm $\beta = \frac{q}{f}$ for some $f > 1$:

- Number of iterations r
- Moduli p_i
- Block sizes b_i
- List size $|L_i| = p_i^{b_i}$

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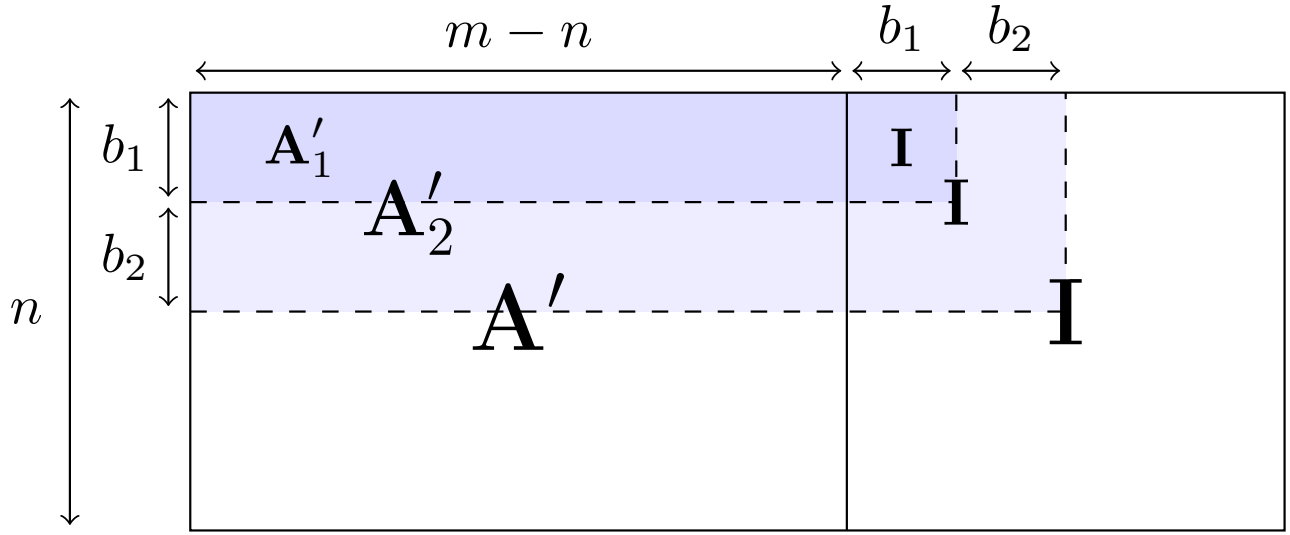

Parameter selection for target norm $\beta = \frac{q}{f}$ for some $f > 1$:

- Number of iterations $r = \log_2 \beta - 1$
- Moduli $p_i = q/2^i$
- Block sizes $b_i = \frac{\ln N}{\ln p_i}$ so that $N = |L_i| = p_i^{b_i}$
- List size $|L_i| = N$ such that it ensures

$$n = \sum_{i=1}^{r} b_i$$

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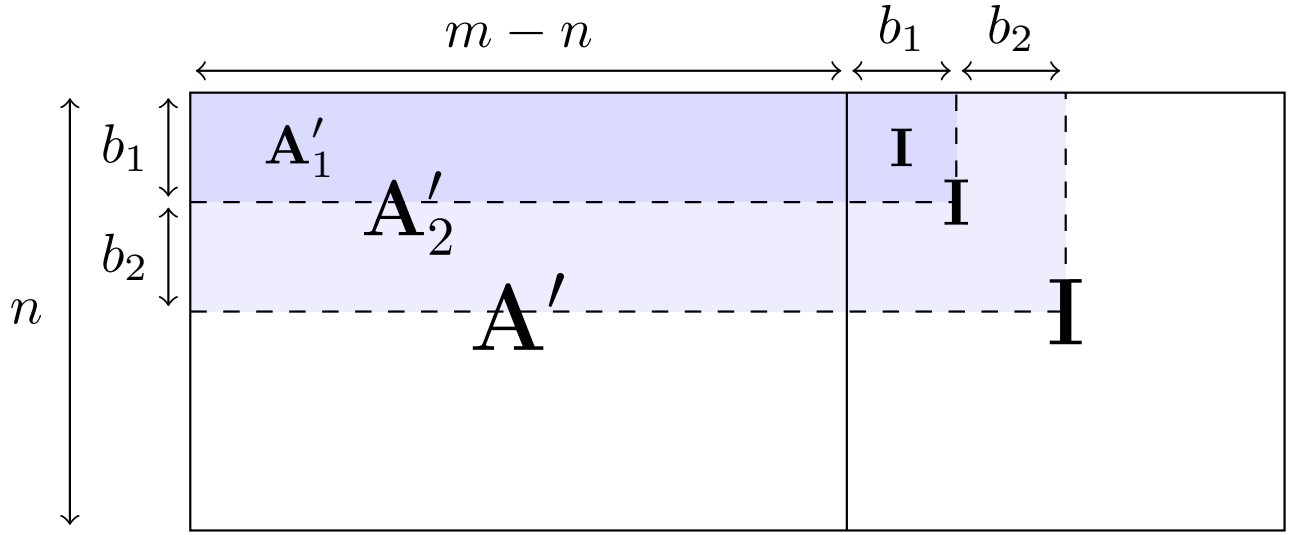

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$$n = \sum_{i=1}^{r} b_i \leq \int_1^{r+1} b_x \mathrm{d}x \leq \log_2(N) \cdot (\ln \ln q - \ln \ln f)$$

Kirchner-Fouque algorithm

Wagner step in Kirchner-Fouque [KF15]

For $n, m \in \mathbb{N}$, $q = \mathrm{poly}(n)$ and $f > 1$, $\beta := \frac{q}{f}$, we are given $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$.
There exists an algorithm that returns a vector $\mathbf{x} \in \mathbb{Z}_q^m$ such that

- $\mathbf{A}\mathbf{x} = \mathbf{0} \bmod q$
- $\|\mathbf{x}\|_\infty \leq \beta = \frac{q}{f}$

in time

$$r \cdot N = \mathrm{poly}(n, \log q) \cdot 2^{\frac{n}{\ln\ln(q) - \ln\ln(f)}}$$

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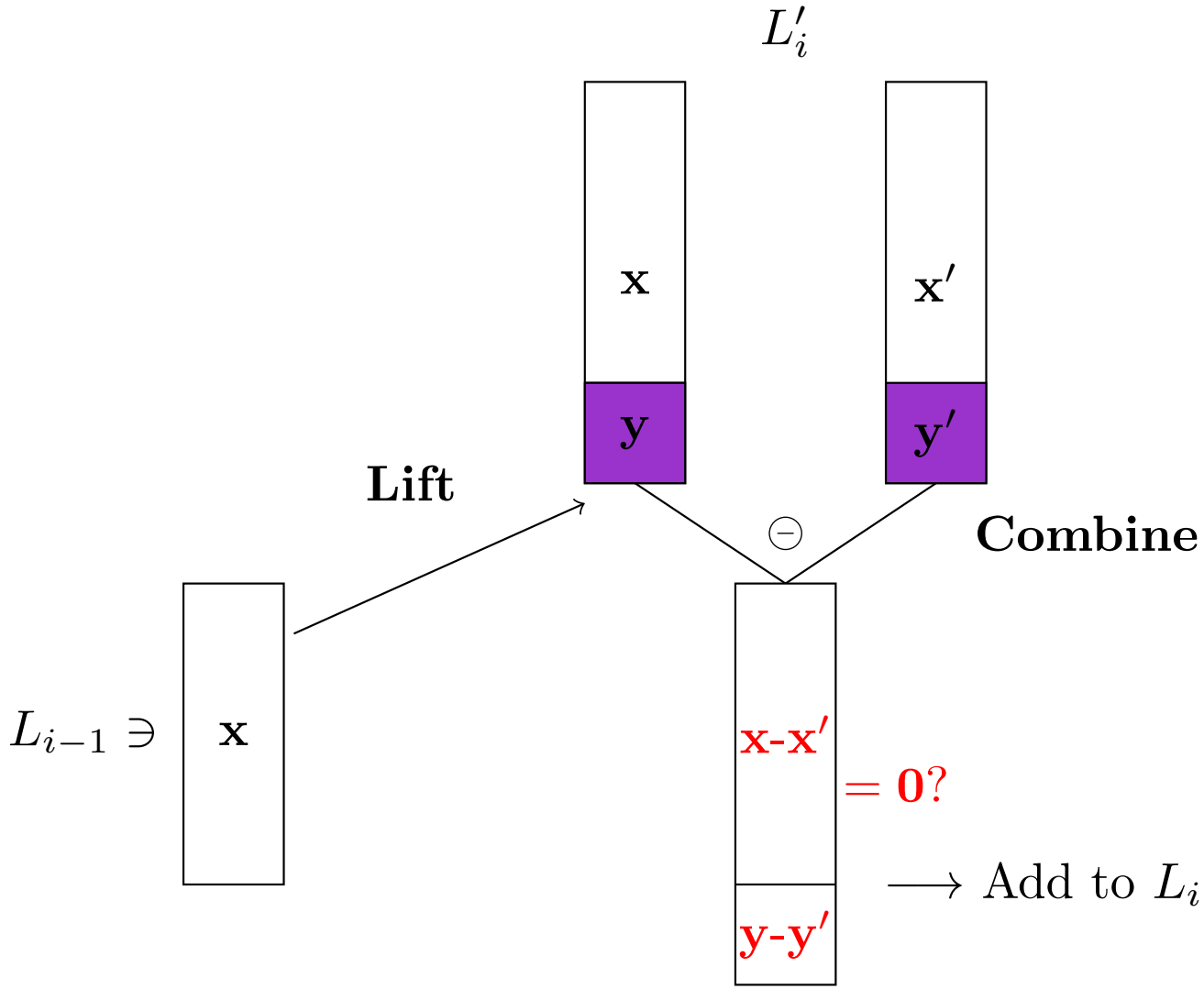
- $\|\mathbf{x}\|_\infty \leq \beta = \frac{q}{f}$

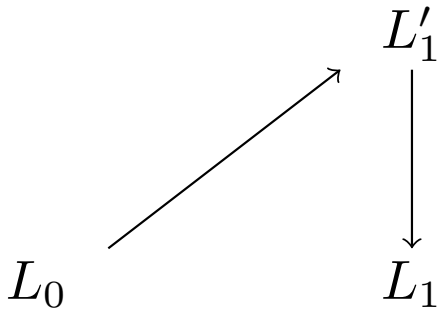
in time

$$r \cdot N = \text{poly}(n, \log q) \cdot 2^{\frac{n}{\ln\ln(q) - \ln\ln(f)}}$$

Is $\mathbf{x}$ non-zero?


L'_i
x
x′
y
y′
Lift
⊖
Combine
$L_{i-1} \ni$
x
x-x′
= 0?
⟶ Add to L_i
y-y′

Distributions of the lists?



$\mathcal{U}^N(\{-1,0,1\}^{m-n})$

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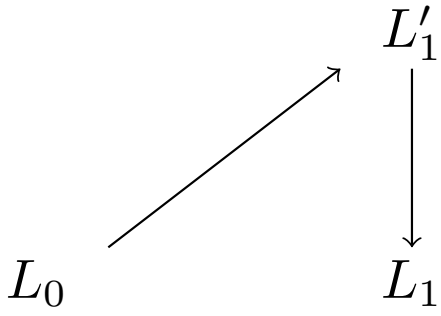
$|\{-1,0,1\}^{m-n}| \gg |\mathbb{Z}_q^{b_1}|$



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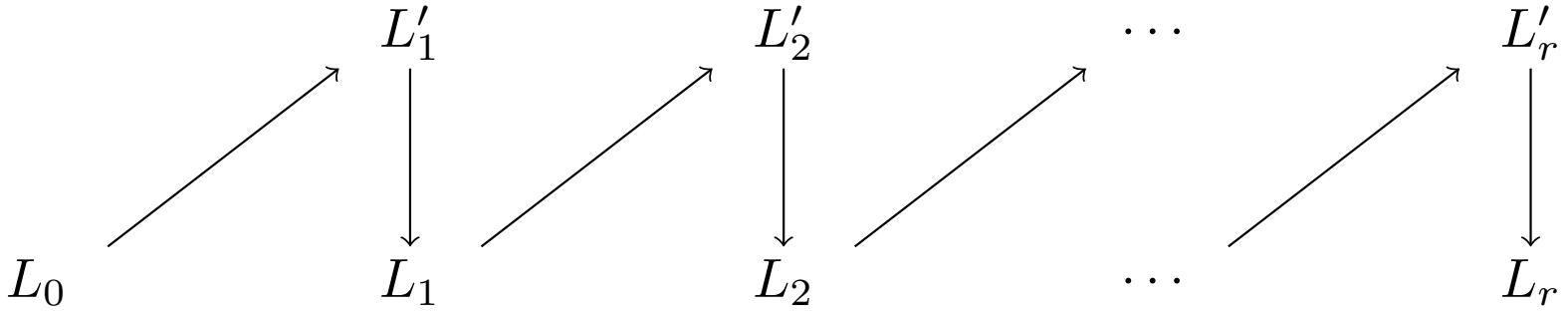

$\mathcal{U}^N(\{-1,0,1\}^{m-n}) \quad \neq \{\mathbf{0}\} \qquad ? \qquad ???$

Outline

$\mathrm{SIS}^{\infty}_{n,m,q,\beta}$ matrix $\mathbf{A}$

Given $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, find a non-zero vector $\mathbf{x} \in \mathbb{Z}_q^m$ such that

- $\mathbf{A}\mathbf{x} = \mathbf{0} \mod q$
- $\|\mathbf{x}\|_\infty \leq \beta$

$$\Longleftrightarrow$$

β-SVP^{∞} in the lattice $\Lambda_q^{\perp}(\mathbf{A})$

Given $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, find a non-zero vector $\mathbf{x} \in \mathbb{Z}_q^m$ such that

- $\mathbf{x} \in \Lambda_q^{\perp}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{Z}^m : \mathbf{A}\mathbf{x} = \mathbf{0} \bmod q\}$
- $\|\mathbf{x}\|_\infty \leq \beta$

$$\Lambda_i := \Lambda_q^{\perp}(\mathbf{A}_i) = \{\mathbf{x} \in \mathbb{Z}^{m-n+n_i} : \mathbf{A}_i\mathbf{x} = \mathbf{0} \bmod q\} = \mathcal{L}(\mathbf{B}_i)$$

$$\Lambda'_i := \mathcal{L}(\mathbf{B}'_i), \text{ with } \mathbf{B}'_i := \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_{m-n} \\ \mathbf{0} & q\mathbf{I}_{n_{i-1}} & \boxed{-\mathbf{A}'_i} \\ \frac{q}{p_i}\mathbf{I}_{b_i} & \mathbf{0} & \end{pmatrix} =: \mathbf{B}_{i-1}$$

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$$\mathbb{Z}^{m-n} = \Lambda_0 \twoheadleftarrow \Lambda_1 \twoheadleftarrow \Lambda_2 \twoheadleftarrow \cdots \twoheadleftarrow \Lambda_r = \Lambda_q^{\perp}(\mathbf{A})$$

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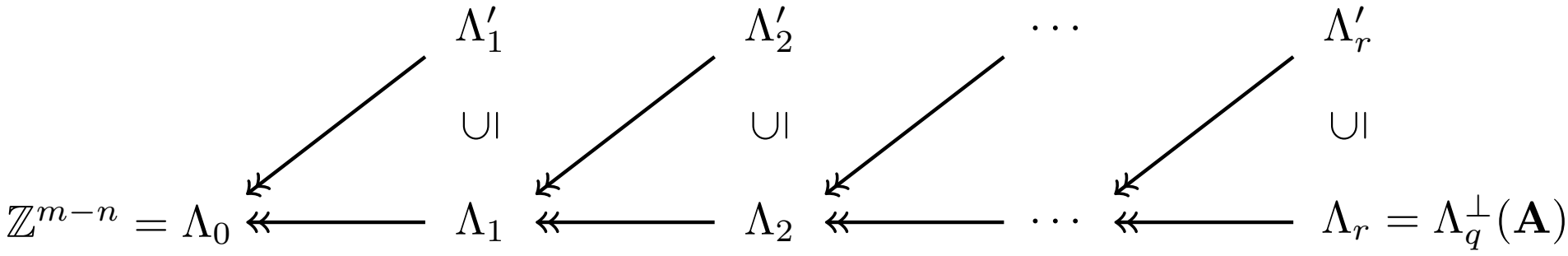

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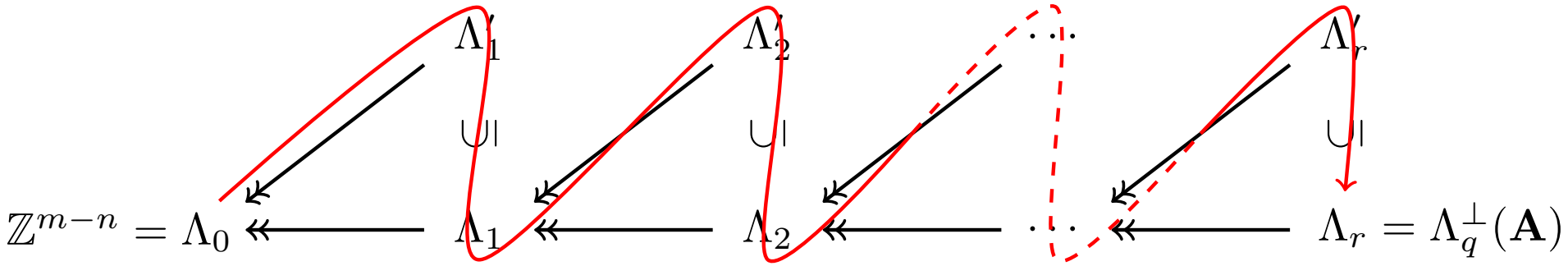


Discrete Gaussian distribution

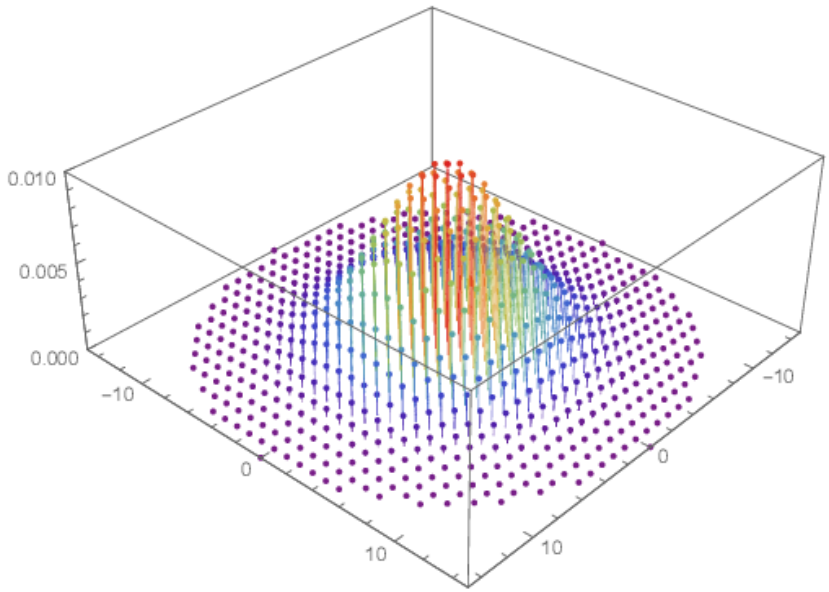
For any $s > 0$ and $\mathbf{x} \in \mathbb{R}^n$, the Gaussian function is $\rho_s(\mathbf{x}) := e^{-\pi(\|\mathbf{x}\|_2/s)^2}$.

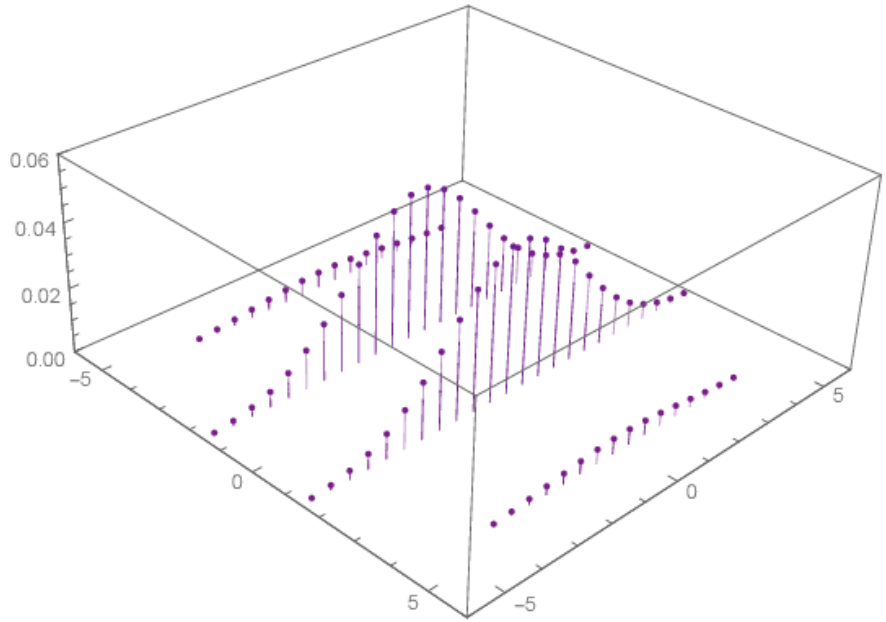
$$\rho_s(\mathcal{L}) := \sum_{\mathbf{x}\in\mathcal{L}} \rho_s(\mathbf{x})$$

Discrete Gaussian distribution

For a full-rank lattice $\mathcal{L}$ and any $s > 0$, the discrete Gaussian distribution $D_{\mathcal{L},s}$ is defined by

$$\Pr_{X\sim D_{\mathcal{L},s}}[X = \mathbf{x}] = \frac{\rho_s(\mathbf{x})}{\rho_s(\mathcal{L})}.$$





Discrete Gaussian distribution

Convolution lemma

Let be a lattice $\mathcal{L} \subseteq \mathbb{R}^n$, $\varepsilon > 0$ and $s \geq \eta_\varepsilon(\mathcal{L})$. For $X_1, X_2 \sim D_{\mathcal{L},s}$,

$$X_1 - X_2 \sim_{3\varepsilon} D_{\mathcal{L},\sqrt{2}s}.$$

Smoothing parameter: $\eta_\varepsilon(\mathcal{L}) := \inf\{s > 0 : \rho_{1/s}(\mathcal{L}^* \setminus \{\mathbf{0}\}) \leq \varepsilon\}$.

Lower bound on s for $D_{\mathcal{L},s}$ to 'behave like' a continuous Gaussian distribution.

Wagner as a Gaussian sampler

Wagner-style algorithm for SIS^∞ [our algorithm]

Input: $\mathbf{A} = [\mathbf{A}' \mid \mathbf{I}_n] \in \mathbb{Z}_q^{n \times m}$
Output: List of vectors $\mathbf{x} \in \Lambda_q^\perp(\mathbf{A})$

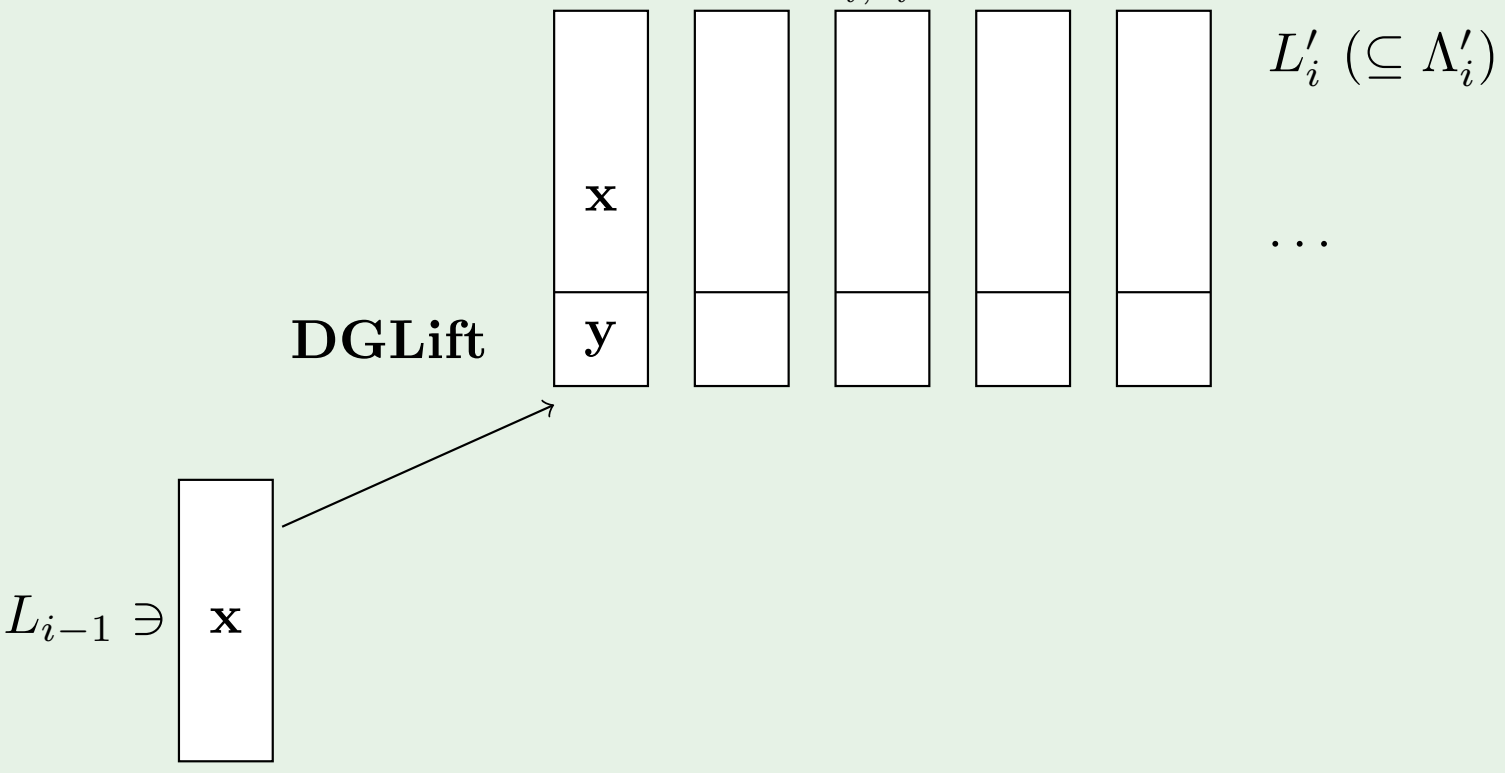
Define sequences of lattices Λ_i, Λ'_i for $i = 0, \ldots, r$
Set s_0 such that all the $s_i := \sqrt{2}^i s_0$ satisfy the smoothness conditions
Initialize a list L_0 with vectors following D_{Λ_0, s_0}
for $i = 1, \ldots, r$ **do**
    $L_i := \mathbf{LiftAndCombine}(L_{i-1}, \Lambda_i)$ $\quad \triangleright \forall \mathbf{x} \in L_i, \mathbf{x} \sim_\varepsilon D_{\Lambda_i, s_i}$
return L_r

Wagner as a Gaussian sampler

LiftAndCombine [our algorithm]

Input: List L_{i-1} of vectors $\mathbf{x} \sim D_{\Lambda_{i-1}, s_{i-1}}$

Output: List L_i of vectors $\mathbf{x} \sim D_{\Lambda_i, s_i}$



Wagner as a Gaussian sampler

LiftAndCombine [our algorithm]

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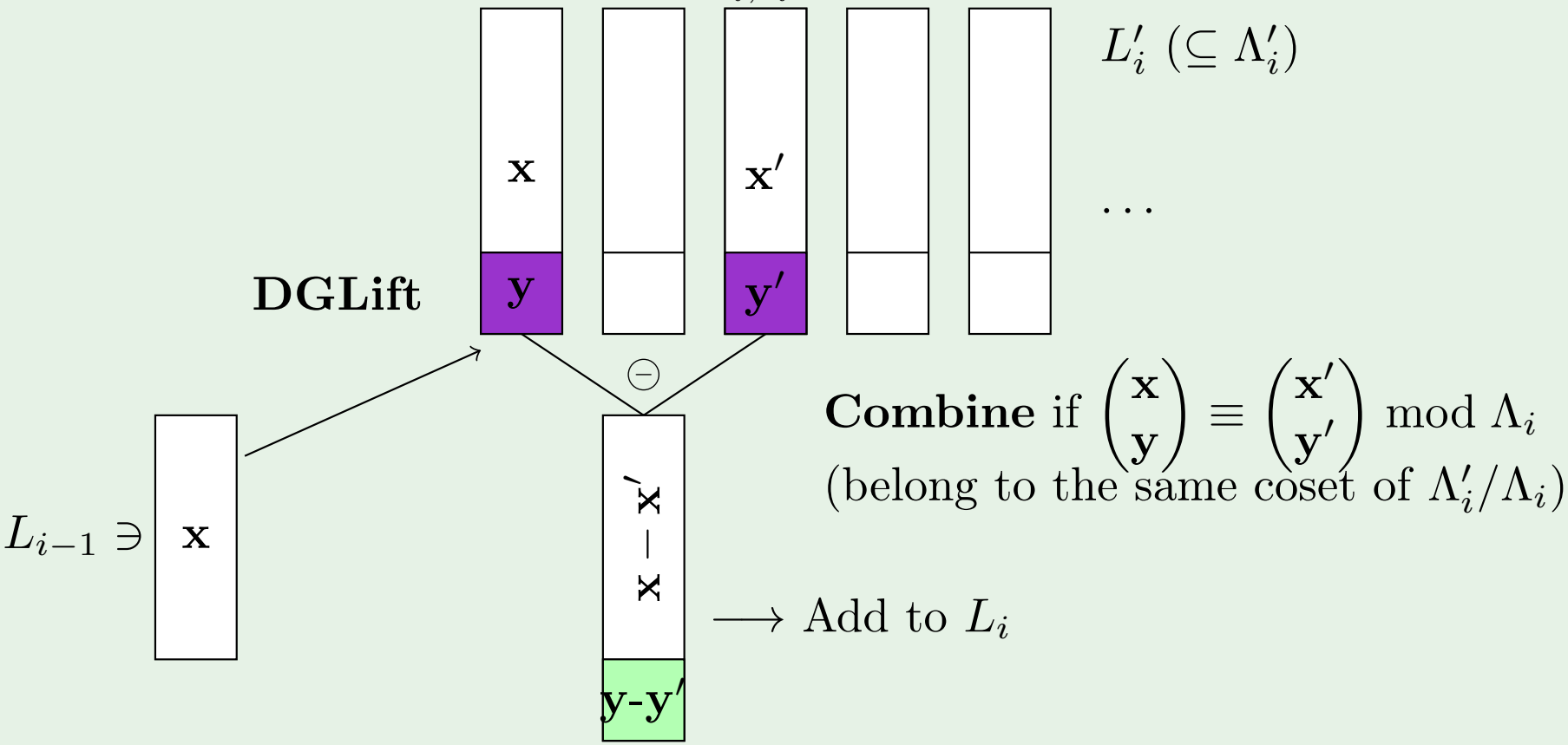

Combine if $\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \equiv \begin{pmatrix}\mathbf{x}'\\ \mathbf{y}'\end{pmatrix} \bmod \Lambda_i$
(belong to the same coset of Λ'_i/Λ_i)

Outline

Implications for SIS

Main theorem: Solving SIS^∞ in provable subexponential time

- $n \in \mathbb{N}$
- $m = n + \omega(n/\log\log n) \in \mathbb{N}$
- $q = \mathrm{poly}(n)$ prime such that $q^{1-n/m} \geq 6$
- $0 \leq \varepsilon \leq \frac{1}{mq^2}$
- $f > 1$ such that $\frac{q}{f} \geq \sqrt{\ln(1/\varepsilon)}$
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There exists an algorithm that solves $\mathrm{SIS}^\infty_{n,m,q,\beta}$ in expected time

$$T = \mathrm{poly}(n, \ln(1/\varepsilon)) \cdot 2^{\frac{n/2}{\ln\ln q - \ln\left(\ln f + \frac{1}{2}\ln\ln\frac{1}{\varepsilon}\right) - O(1)}} = 2^{O\left(\frac{n}{\ln\ln n}\right)}$$

with success probability $1 - 2^{-\tilde{\Omega}(n)}$.

Implications for ternary-LWE

Definition: Decision-LWE (Learning With Errors)

Let $\mathbf{s} \sim \mathcal{U}(\mathbb{Z}_q^n)$ and χ be a probability distribution on $\mathbb{Z}$. Decide whether given pairs $(\mathbf{a}, c) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$ are sampled according to

- the uniform distribution on $\mathbb{Z}_q^n \times \mathbb{Z}_q$; or
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Work in progress:

- Choose a sequence of lattices with smaller η_ϵ (than $\frac{q}{p_i}\mathbb{Z}^{b_i}$)
- Dual distinguisher by statistical decoding [CDMT22]

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In practice, attacks run faster than the proven version.
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NIST level	n	m	q	β	$\log_2$(Time)
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This attack does not seem to threaten Dilithium

Conclusion

- A *provable* algorithm for SIS^∞ in subexponential time $2^{O\left(\frac{n}{\ln \ln n}\right)}$
- Leads for getting a similar result for LWE
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Take-away

× Work with coordinates of vectors, roundings, parity-check matrices...

✓ Explicit the mathematical structures underlying the problems

Research directions

- Prove an algorithm for LWE in subexponential time
  - Select a sequence of lattices with smaller η_ϵ (than $\frac{q}{p_i}\mathbb{Z}^{b_i}$)
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Thank you for your attention!

References

[AFFP14] Martin R Albrecht et al. "Lazy modulus switching for the BKW algorithm on LWE." In: *PKC*. 2014.

[AR05] Dorit Aharonov and Oded Regev. "Lattice problems in NP ∩ coNP." In: *ACM* (2005).

[BKW03] Avrim Blum, Adam Kalai, and Hal Wasserman. "Noise-tolerant learning, the parity problem, and the statistical query model." In: *ACM* (2003).

[CDMT22] Kevin Carrier et al. "Statistical decoding 2.0: Reducing decoding to LPN." In: *Asiacrypt*. 2022.

[HKM18] Gottfried Herold, Elena Kirshanova, and Alexander May. "On the asymptotic complexity of solving LWE." In: *Designs, Codes and Cryptography* (2018).

[KF15] Paul Kirchner and Pierre-Alain Fouque. "An improved BKW algorithm for LWE with applications to cryptography and lattices." In: *CRYPTO*. 2015.