Modelling Power Noise via Gaussian Processes with Applications to True Random Number Generators

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Outline

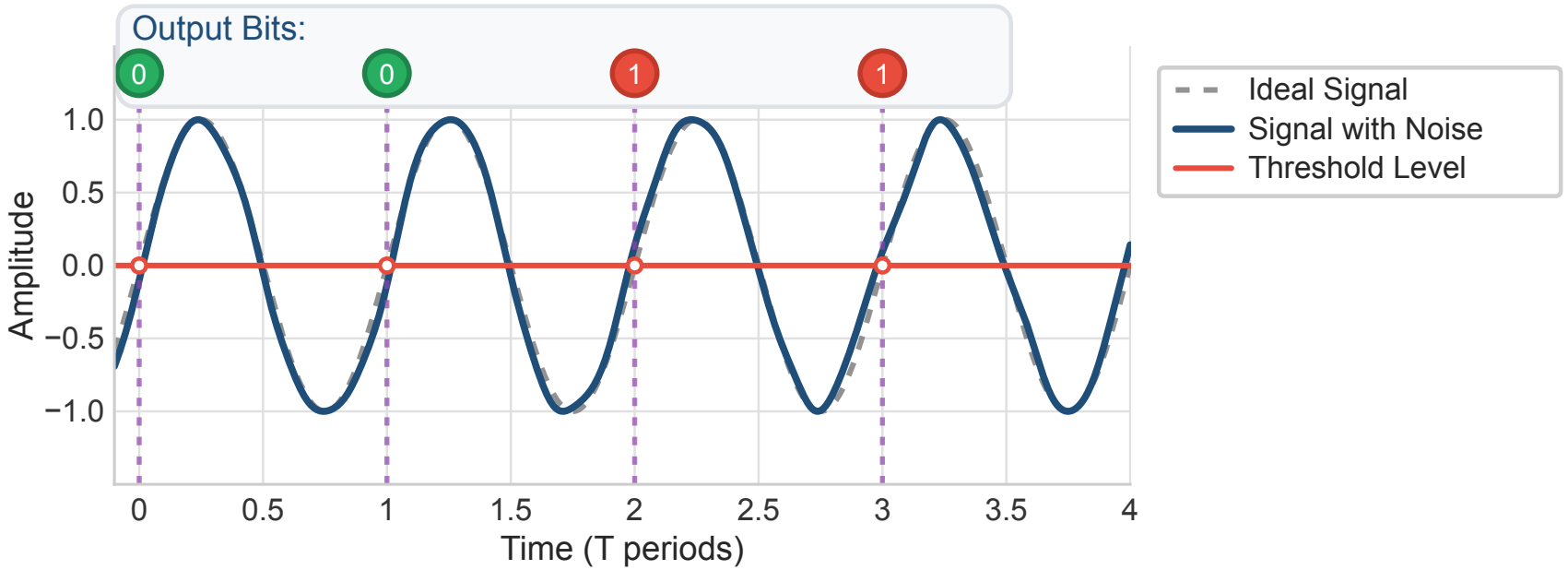
Oscillatory Random Number Generators

- Leverage physical noise in electronic circuits for cryptography
- Generate bits by thresholding instantaneous phase:

$$b_t = w(2\pi f_0 t + \phi_0 + \phi_t)$$

where w is a square wave with duty cycle α and ϕ_t is phase noise.

- Randomness due to signal crossings earlier or later than expected



Modelling Challenge

- Electronic principles and experiments provide understanding of spectrum...

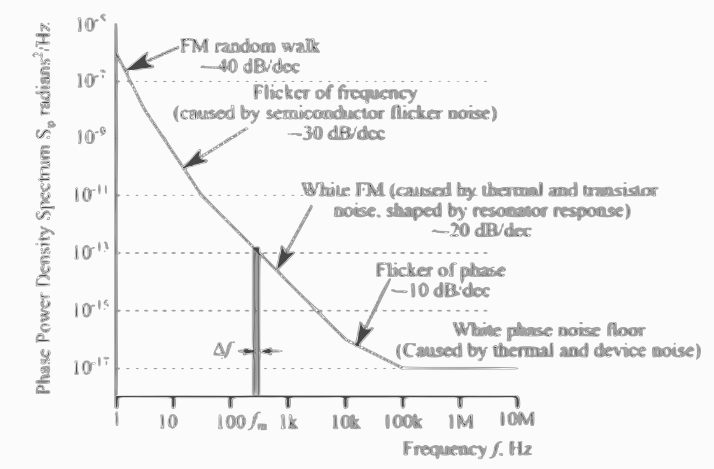


Figure: Power Law for phase spectrum.

Leeson's model of noise in oscillators [Lee66]

$$S_\phi(\omega) = \frac{2FkT}{P_s}\left(1+\left(\frac{\omega_0}{2Q_L\omega}\right)^2\right)\left(1+\frac{\omega_c}{\omega}\right)$$

where $S_\phi(\omega)$ is phase noise PSD, F is amplifier noise factor, k is Boltzmann's constant, T is temperature, P_s is signal power, ω_0 is carrier frequency, Q_L is loaded quality factor, ω_m is offset frequency, and ω_c is flicker corner frequency,...

- **Cryptographic security** requires *time-domain* properties, particularly conditional dependencies / predictability under partial observation (leakage)
- **Gap:** Spectral models $\neq$ temporal correlation models

Related Work

- Basic properties of 1/f noise correlations studied [Kes82]
- Gaussian process model for 1/f noise proposed [BJTA82], but without accurate temporal and spectral analysis
- Hardware implementations shown randomness quality gains from 1/f noise [BCF+24]
- Short-memory numerical approximations proposed for TRNG simulation [PV24]
- **Gap:** No tractable analytical model capturing long-range dependencies in coloured noise for security analysis

Novel Contributions

1. **Realistic Noise Model**: Fractional Brownian Motion for phase noise

$$\phi_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-u)^{H-1/2} dB_u$$

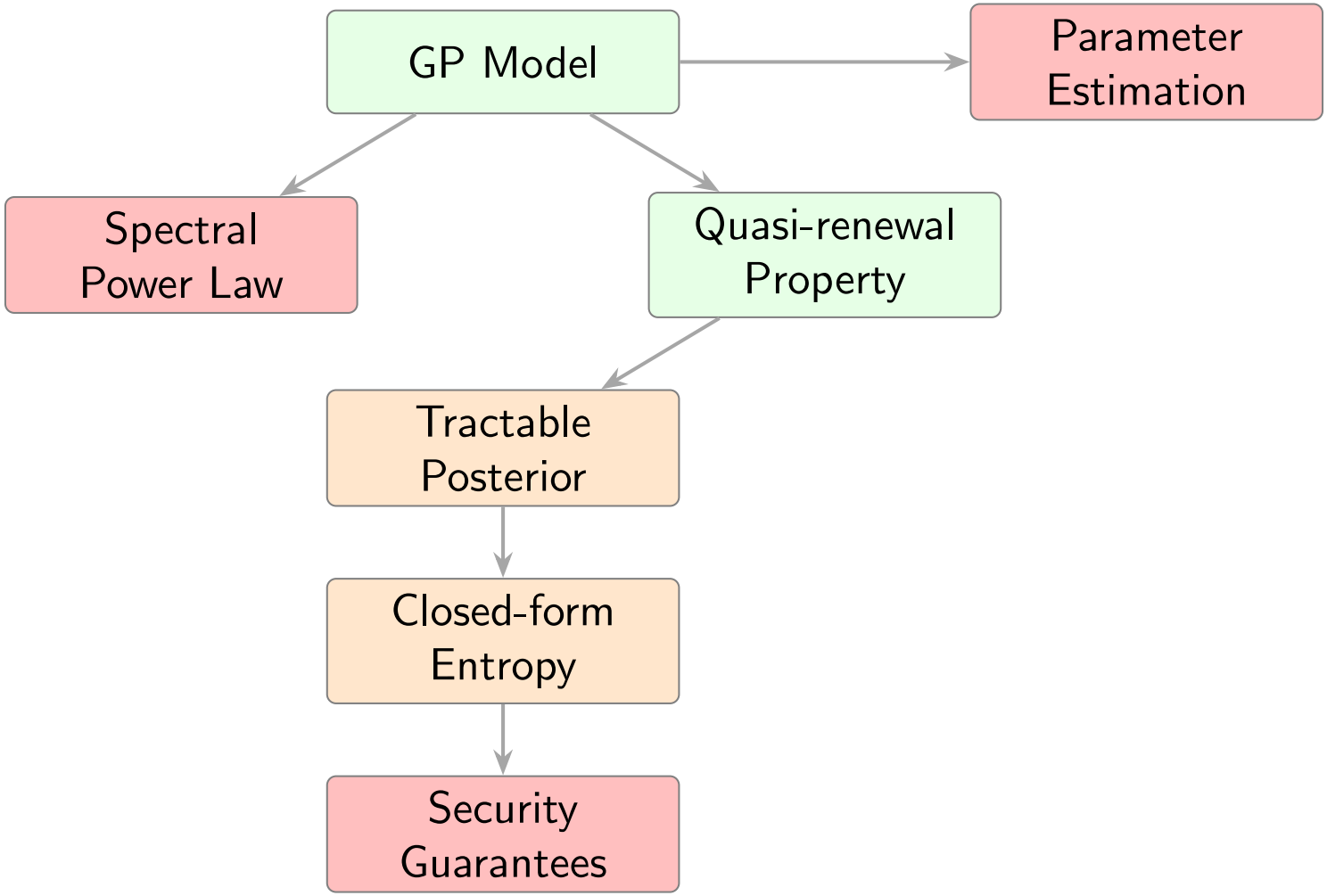
   with spectrum $\propto \omega^{-2H-1}$
2. **Quantified Leakage Resilience**: Conditional variance scales as τ^{2H}
3. **Closed-Form Entropy Analysis**: Exact entropy of Gaussian posteriors
4. **Parameter Estimation**: Allan variance-based coefficient recovery

Key Achievement

First tractable analytical model bridging power-law spectra ω^{-2H-1} (white to flicker noise) with cryptographic security requirements

Preprint available: [Sko24]

Theoretical Framework



Fractional Brownian Motion Properties

Using basics of Ito calcuus one obtains:

Covariance (Closed Form)

$$\mathrm{Cov}(\phi_s^H, \phi_t^H) = \frac{t^{H-\frac{1}{2}} s^{H+\frac{1}{2}}\, {}_2F_1\left(1, \frac{1}{2} - H; H + \frac{3}{2}; \frac{s}{t}\right)}{\Gamma(H + \frac{1}{2})\Gamma(H + \frac{3}{2})}, \quad 0 < s < t$$

Variance

$$\mathrm{Var}[\phi_t^H] = \frac{t^{2H}}{2H\Gamma(H + \frac{1}{2})^2}, \quad 0 < t$$

Correlation (Time Ratio Dependent)

$$\mathrm{Cor}(\phi_t^H, \phi_s^H) = \frac{4H\, {}_2F_1\left(1, \frac{1}{2} - H; H + \frac{3}{2}; \frac{1}{r}\right)}{(2H+1)\sqrt{r}}$$

where $r = \max\{t, s\}/\min\{t, s\}$.

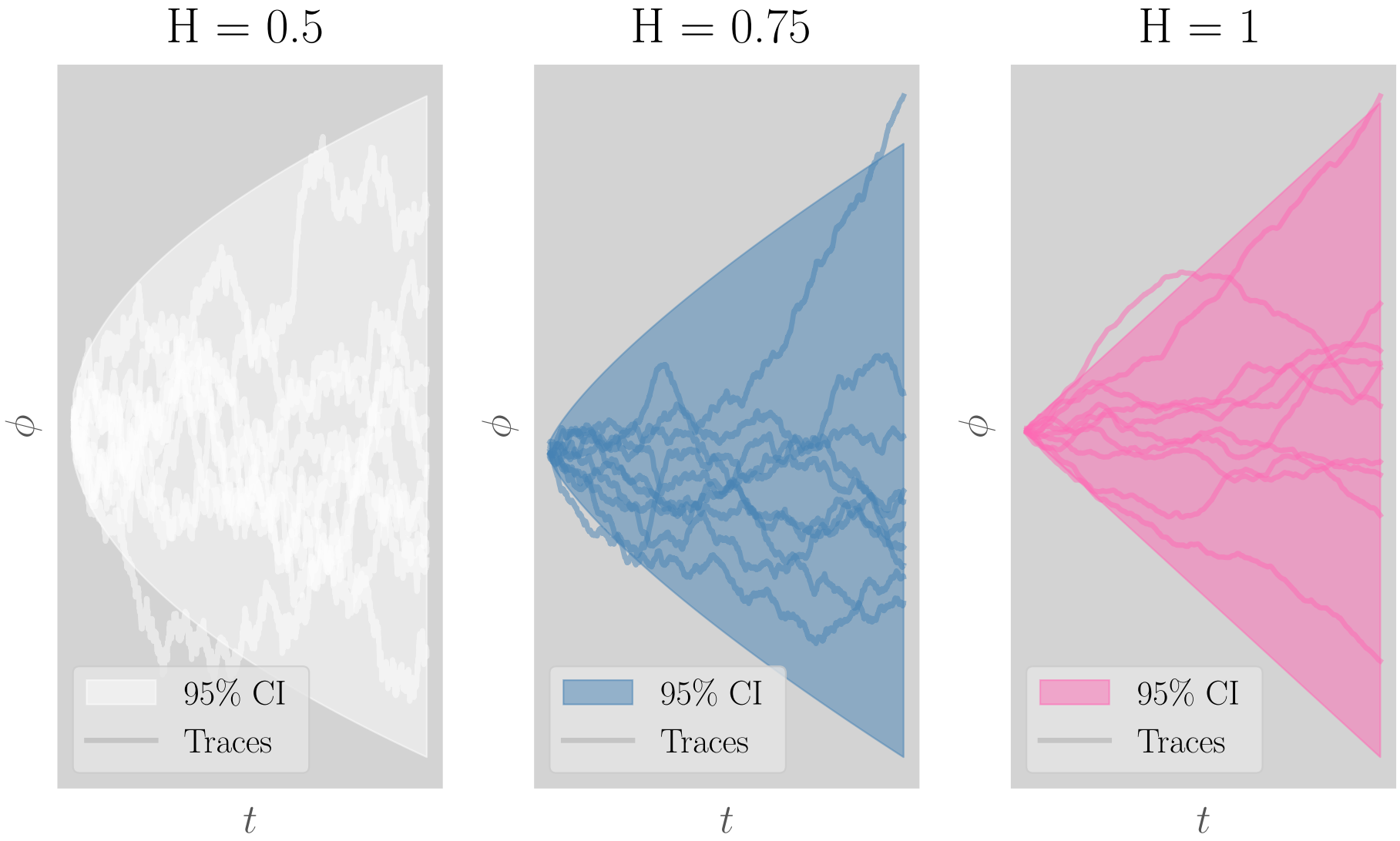


Figure: Sample paths for different Hurst parameters.

Spectral Analysis: Theory

Wigner-Ville Spectrum

Time-averaged spectrum exhibits power law:

$$\bar{S}_{\phi_H}(T,\omega) = \omega^{-2H-1}\left(1 + O_H((T\omega)^{H-3/2})\right)$$

White Noise ($H = 0.5$):

$$S_\phi(\omega) \propto \omega^{-2}, \quad S_{\dot{\phi}}(\omega) \propto \omega^0$$

Flicker Noise ($H = 1$):

$$S_\phi(\omega) \propto \omega^{-3}, \quad S_{\dot{\phi}}(\omega) \propto \omega^{-1}$$

where $\dot{\phi} = \frac{\Delta\phi}{\Delta t}$ is instantaneous frequency

Validation

FBM captures dominant TRNG noise sources with exact power laws

Spectral Analysis: Proof

For closed form:

- Express covariance in time-lag coordinates: $K(t_1, t_2) = K(t - \tau, t + \tau)$
- Observe that derivatives are rational functions
- Evaluate Fourier transform of derivatives, then integrate back

For asymptotics:

- Use properties of hypergeometric functions

Spectral Analysis: Empirical Validation

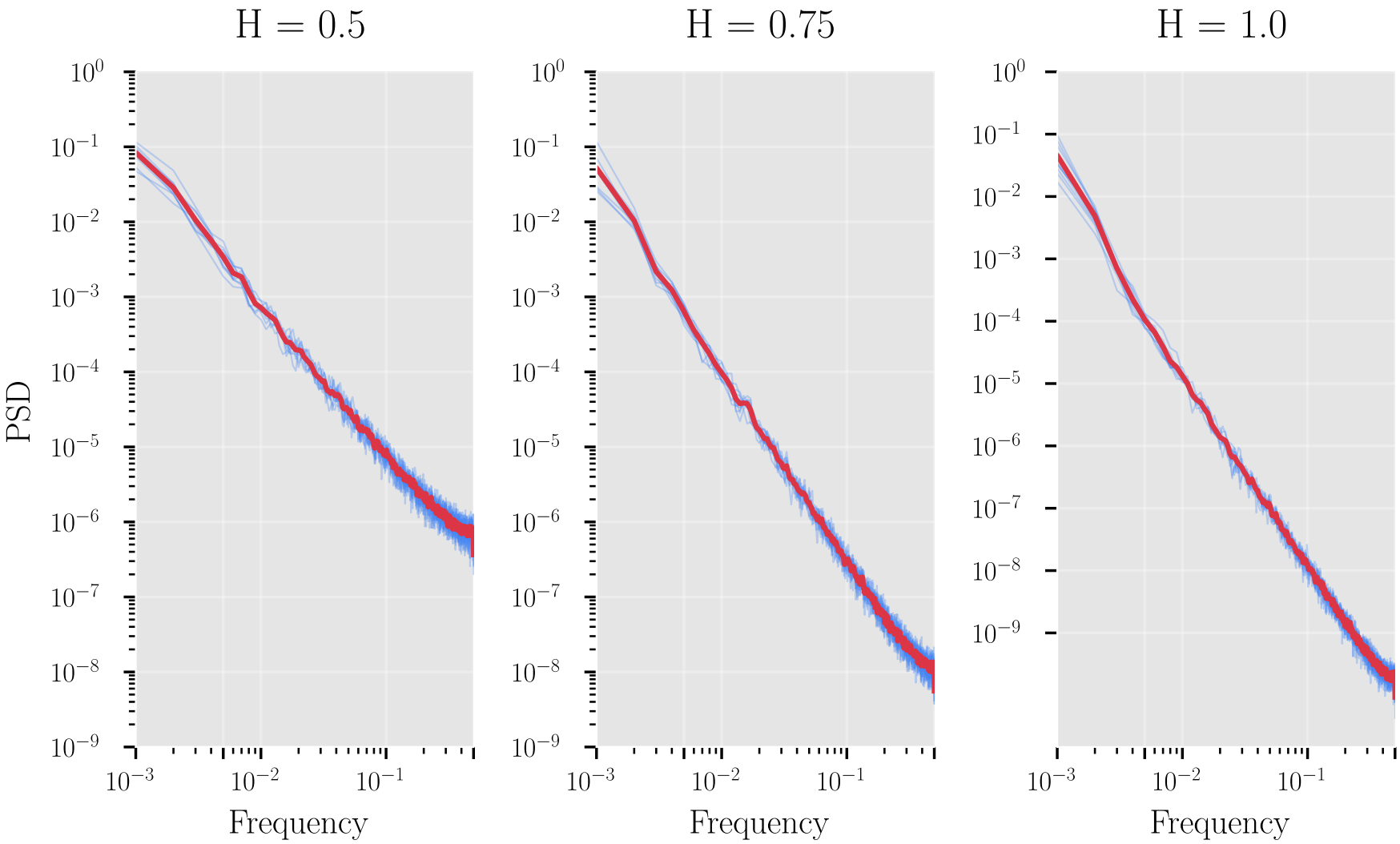


Figure: Power spectral density for various Hurst exponents.

Leakage Resilience

Quasi-Renewal Property

For $s < t$:

$$\mathrm{Var}(\phi_t^H \mid \phi_{\leq s}^H) = \mathrm{Var}(\phi_{t-s}^H) = \frac{(t-s)^{2H}}{2H\Gamma(H+\frac{1}{2})^2}$$

Security implications:

- Conditional variance depends only on time gap $\tau = t - s$
- Uncertainty grows as τ^{2H} (power-law recovery)
- Enables tractable analysis of information leakage
- Process "restarts" after conditioning on past observations

Key Insight

Complete observation history provides significantly more predictive power than pairwise correlations suggest—crucial for TRNG security analysis

Entropy Analysis

Folded / Wrapped Gaussian Distribution

For $Y = X \bmod r$ where $X \sim \mathcal{N}(\mu, \sigma^2)$:

$$p_Y(y) = r^{-1}\vartheta_3\left(\pi(\mu - y)/r, e^{-2\pi^2\sigma^2/r^2}\right)$$

Min-Entropy under Gaussian Posterior

$$\max_{\phi_0} \text{bias}(b_t \mid \Phi_{\leq s}, \phi_0) = \epsilon(\sigma(t|s), \alpha)$$

where

$$\epsilon(\sigma, \alpha) = \left| \frac{1}{\pi} \int_0^{\alpha\pi} \vartheta_3\left(y/2, e^{-\sigma^2/2}\right) dy - \frac{1}{2} \right|$$

Higher noise variance $\rightarrow$ Lower bias $\rightarrow$ Higher entropy

Parameter Estimation

Allan Variance Method

For twice-differenced FBM:

$$\mathrm{Var}(\Delta^2_{t,h}\phi^H_t) = h^{2H}\left(\frac{(4-4^H)\csc(H\pi)}{\Gamma(2H+1)} + O_H((h/t)^{4-2H})\right)$$

Special Cases

$$H = 1/2: \quad \mathrm{Var}(\Delta^2_{t,h}\phi^{1/2}_t) = 2h(1+O((h/t)^3))$$

$$H = 1: \quad \mathrm{Var}(\Delta^2_{t,h}\phi^1_t) = \frac{4\log 2}{\pi}h^2(1+O((h/t)^2))$$

- Nearly unbiased estimation
- Connects theoretical constants to hardware measurements
- Validated with FPGA data (1M samples on CycloneV and SmartFusion)

Sampling Bandwidth

Mixed Noise Model

For phase noise $\phi = \sum_{H \in \{1/2,1\}} c_H \phi_t^H$:

$$\sigma^2 = \sum_{H \in \{1/2,1\}} c_H^2 \frac{(\Delta t)^{2H}}{2H\Gamma(H + \frac{1}{2})^2}$$

Min-Entropy Rate

$$\mathbf{H}_\infty(b_n \mid b_{\leq n-1}, \phi_0) = \epsilon(\sigma, \alpha)$$

Design Implication

Sampling interval Δt directly determines security through noise variance trade-off

Summary & Impact

Achieved

- **Tractable framework** with closed-form expressions
- **Physical realism** via power-law spectral densities
- **Security guarantees** through leakage resilience analysis
- **Calibration** via parameter estimation

Significance

- Bridges physical modelling with cryptographic requirements
- Eliminates simulation-based approach limitations
- Enables rigorous TRNG security evaluation

Future Work

GPU-accelerated sampling for analysing post-processing.

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Thank You

Questions?

References I

- Licinius Benea, Mikael Carmona, Viktor Fischer, Florian Pebay-Peyroula, and Romain Wacquez, *Impact of the Flicker Noise on the Ring Oscillator-based TRNGs*, IACR Transactions on Cryptographic Hardware and Embedded Systems **2024** (2024), no. 2, 870–889.
- JA Barnes, RH Jones, PV Tryon, and DW Allan, *Stochastic models for atomic clocks*, Annual Precise Time and Time Interval Systems and Applications Meeting, 1982, pp. 295–306.
- M.S. Keshner, *1/f noise*, Proceedings of the IEEE **70** (1982), no. 3, 212–218.
- D.B. Leeson, *A simple model of feedback oscillator noise spectrum*, Proceedings of the IEEE **54** (1966), no. 2, 329–330.
- Adriaan Peetermans and Ingrid Verbauwhede, *Trng entropy model in the presence of flicker fm noise*, IACR Transactions on Cryptographic Hardware and Embedded Systems **2024** (2024), no. 4, 285–306.

References II

Maciej Skorski, *Modelling 1/f Noise in TRNGs via Fractional Brownian Motion*, October 2024.