Variations on the Knapsack Generator

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KNAPSACK GENERATOR

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Pseudo Random Number Generator

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Hard computational problem → KNAPSACK

Pseudo Random Number Generator → GENERATOR

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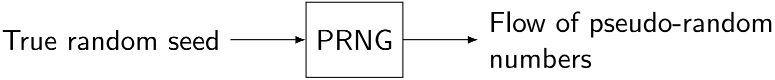
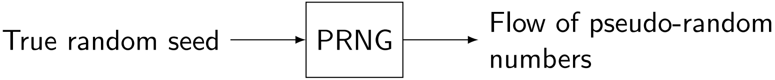
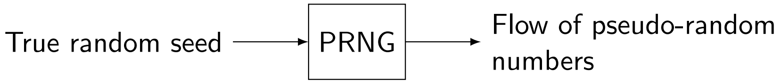

True random seed
PRNG
Flow of pseudo-random numbers

PRNG



- Security is often based on perfect randomness
- "True" randomness is expensive



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- Shared randomness is common in cryptography
- Stream cipher
- Reducing communication in MPC protocols.

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Security of PRNG

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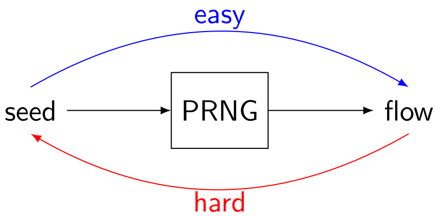
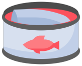
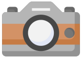


t kg



ω_1 kg



ω_2 kg

ω_3 kg

ω_4 kg

What is in the knapsack ?

Mathematatic version

The weight list:	$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \{0, N\}^n$
The secret composition:	$\mathbf{u} = (u_1, \ldots, u_n) \in \{0, 1\}^n$
The target weight:	$v = \sum \omega_i u_i = \langle \boldsymbol{\omega}, \mathbf{u} \rangle$

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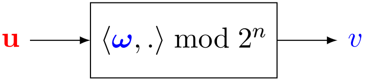
The Subset Sum Problem is NP-hard and remain hard if we replace v by $v \mod N$ as long as $N \simeq 2^n$.

Knapsack Generator by Rueppel and Massey[1]



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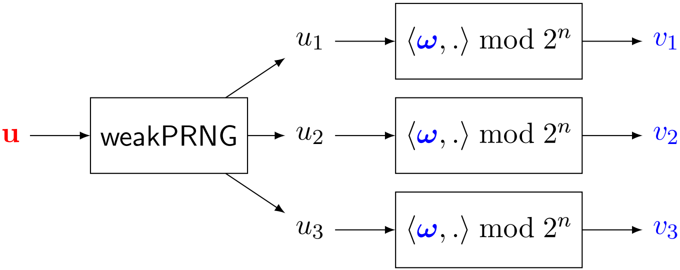

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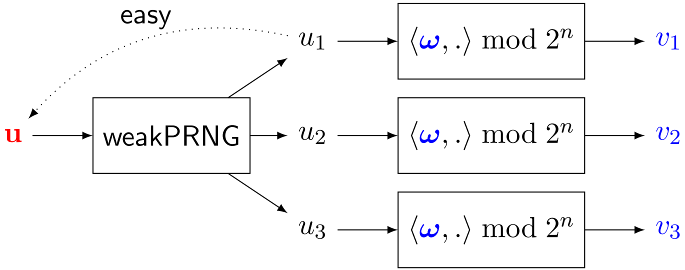

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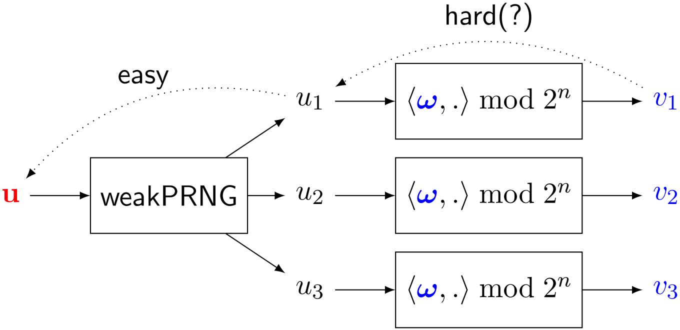

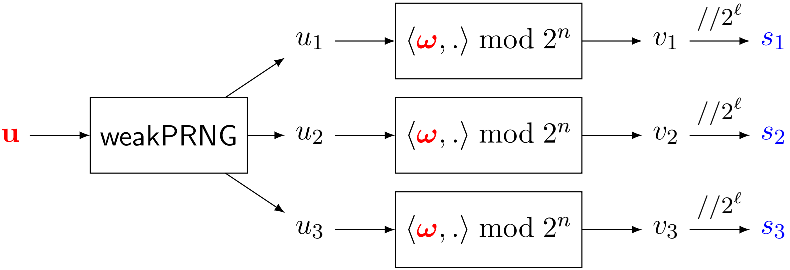

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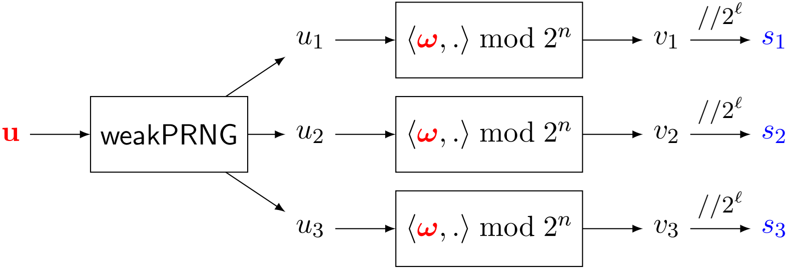

We call δ_i the truncated bits : $v_i = 2^\ell s_i + \delta_i$.

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secret : $\mathsf{u} \quad + \quad \omega$

secret :	u	$+$	$\boldsymbol{\omega}$
	n bits		n^2 bits

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Can we distinguish between the u ?

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Can we distinguish between the u ?
Yes, with OmegaRetriever

We consider m outputs and $\mathbf{s} = (s_1, \ldots, s_m)$.

$$\textsc{OmegaRetriever}: \quad \begin{array}{lll} \mathbf{u}, \mathbf{s} & \to \boldsymbol{\omega}' & \text{close to } \boldsymbol{\omega} \\ \mathbf{u}', \mathbf{s} & \to \boldsymbol{\omega}'' & \text{not close to } \boldsymbol{\omega} \end{array}$$

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$\textsc{KnapsackGen}(u, \boldsymbol{\omega}')$ will be close to $\textsc{KnapsackGen}(u, \boldsymbol{\omega})$.
$\textsc{KnapsackGen}(u', \boldsymbol{\omega}'')$ will be not.

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OmegaRetriever from FSE 2011 (part 2)

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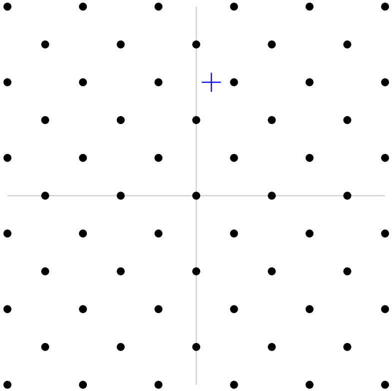
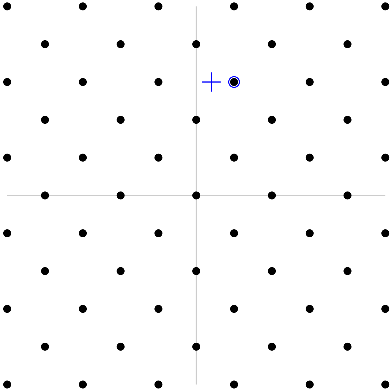
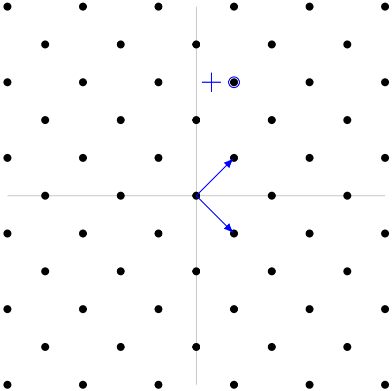
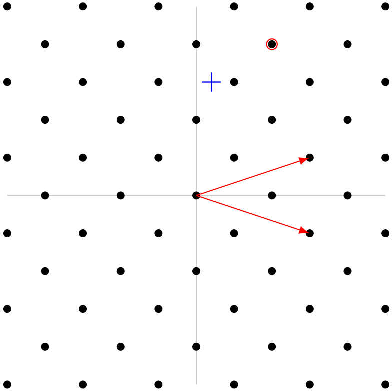
Experimental results are close to the bound.









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But $\boldsymbol{\omega}'$ defined as $U \times \boldsymbol{\omega}' \equiv \mathbf{v}' \bmod 2^n$ is close to $\boldsymbol{\omega}$!

Why does it work ?

- $\mathbf{v} - \mathbf{v}'$ is small and equal to $U \times (\boldsymbol{\omega} - \boldsymbol{\omega}') \bmod 2^n$
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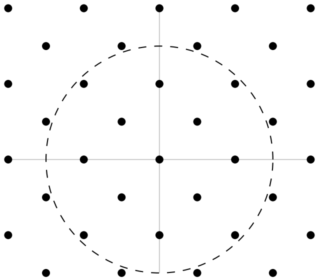
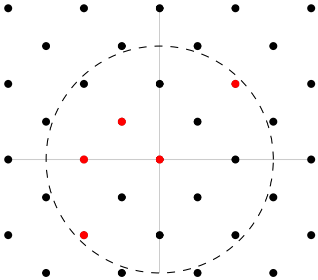
- $\boldsymbol{\omega} - \boldsymbol{\omega}' = T \times (\mathbf{v} - \mathbf{v}') \bmod 2^n$
- We can bound T and $(\mathbf{v} - \mathbf{v}')$
- BUT $\|\boldsymbol{\omega} - \boldsymbol{\omega}'\| \ll \|T\| \times \|(\mathbf{v} - \mathbf{v}')\|$

A Second Idea

1. We know that $\mathbf{v} - \mathbf{v}'$ is small ($\leq K$) and in Λ.



1. We know that $\mathbf{v} - \mathbf{v}'$ is small ($\leq K$) and in Λ.
2. If $\|x\| < K/\|U\|$, then $\|Ux\| < K$.



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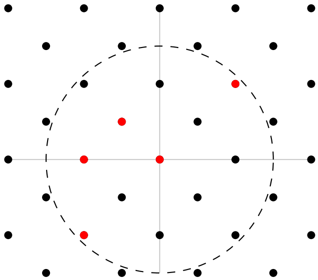
3. How do I know that $(\mathbf{v} - \mathbf{v}')$ is a red point ?



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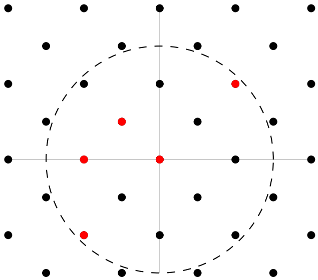

We denote A_K the set of red points

$$|A_K| = (2 \times \lfloor K/\|U\| \rfloor - 1)^n$$

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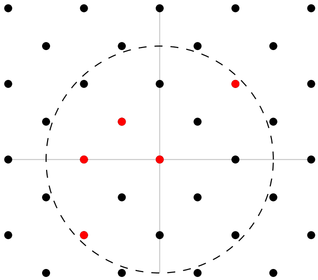

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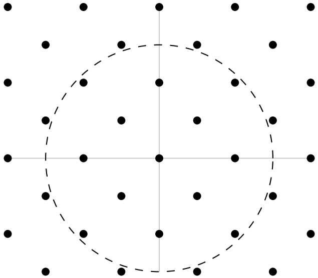
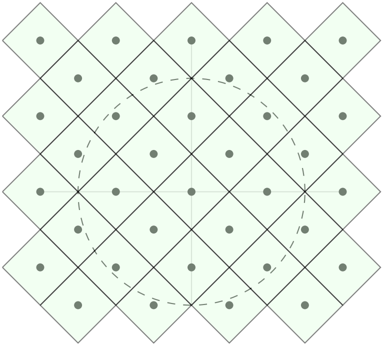
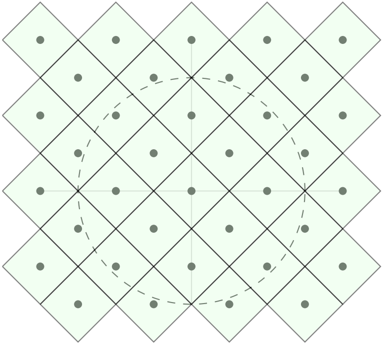
We denote B_K the set of points in the ball

How many point in B_K ?



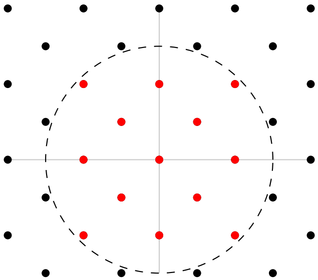




Gaussian Heuristic : $|B_K| \simeq Volume(Ball)/Volume(\Lambda)$

In the case where $n = 32$, $m = 42$ and $\ell \leq 15$,

$$|A_K| \geq |B_K| \text{ with } K = 2^{\ell+1}$$



Thus $\mathbf{v} - \mathbf{v}'$ is a red point and $\|\boldsymbol{\omega} - \boldsymbol{\omega}'\| < K/\|U\|$.

ℓ	5		10		15		20		25	
m	34	40	34	40	34	40	35	40	39	40
✓bits (over 32)	27	28	22	23	5	18	4	13	6	8

Figure: Quality of $\boldsymbol{\omega'}$ for $n = 32$

ℓ	5		10		15		20
m	34	40	35	40	36	40	41
✓bits (over 32)	10	22	10	17	8	12	6

Figure: Quality of $\boldsymbol{\omega'}$ for $n = 32$ for FSE 2011 algorithm

Thank you for your attention,