On cycles of pairing-friendly abelian varieties

Based on joint work with Craig Costello and Michael Naehrig

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Motivation

Pairing-based proof systems

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- In 2014, Ben-Sasson, Chiesa, Tromer and Virza realised efficient pairing-based zk-SNARKs with *recursive composition* of proofs.
- The core ingredient are *cycles of pairing-friendly elliptic curves*.

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\begin{tikzcd}
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Pairing-friendly cycles of elliptic curves

Definition [Ben-Sasson, Chiesa, Tromer, Virza '14]

A *(2-)cycle of pairing-friendly elliptic curves* is a pair of elliptic curves

$$E/\mathbb{F}_p \quad \text{and} \quad E'/\mathbb{F}_q$$

such that

$$q = \#E(\mathbb{F}_p) \quad \text{and} \quad p = \#E'(\mathbb{F}_q)$$

and E, E' are pairing-friendly.

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Main example of a cycle

Example: 2-cycle of Miyaji, Nakabayashi and Takano (MNT) curves

$$(k, k') = (4,6)$$

$$\begin{tikzcd}
\mathbb{F}_{p^4}^{\times} & \mathbb{F}_{q^6}^{\times} \\
\#E(\mathbb{F}_p) = q \arrow[u] \arrow[r, bend left] & \#E'(\mathbb{F}_q) = p \arrow[u] \arrow[l, bend left]
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- MNT cycles known prior to pairing-based proof popularity [Karabina, Teske ’08]

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To overcome these impossibility results, previous works relax *pairing-friendly* or look at *2-chains*.

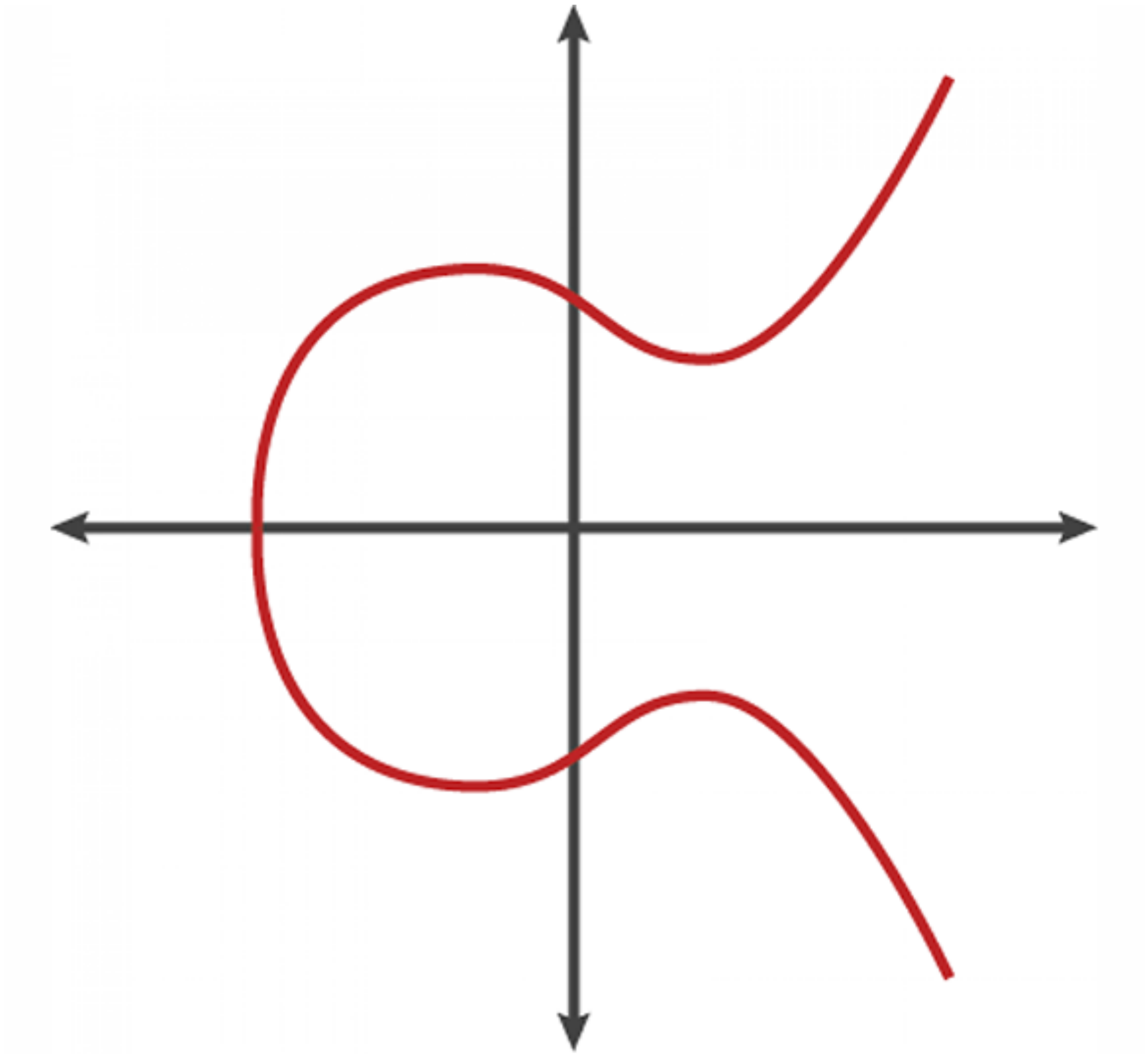
(Short) Preliminaries

Elliptic Curves for Cryptography

An **elliptic curve** E over finite field $\mathbb{F}_q$ ($q = p^{\bullet}, p \neq 2,3$) is a smooth curve

$$E : y^2 = x^3 + ax + b,$$

where $a, b \in \mathbb{F}_q$.



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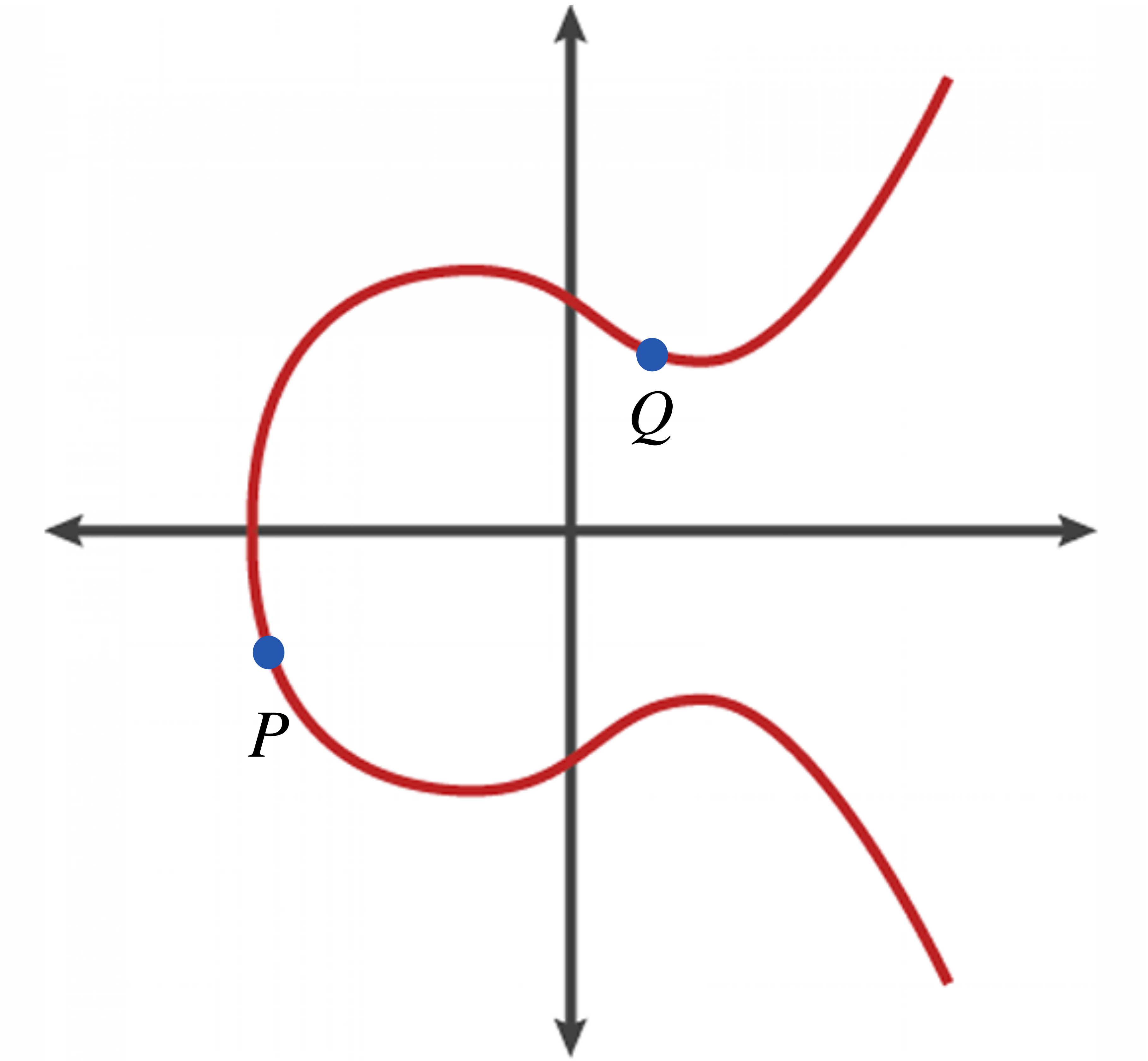
Elliptic curves form a group $E(\mathbb{F}_q)$ under **addition of poi**



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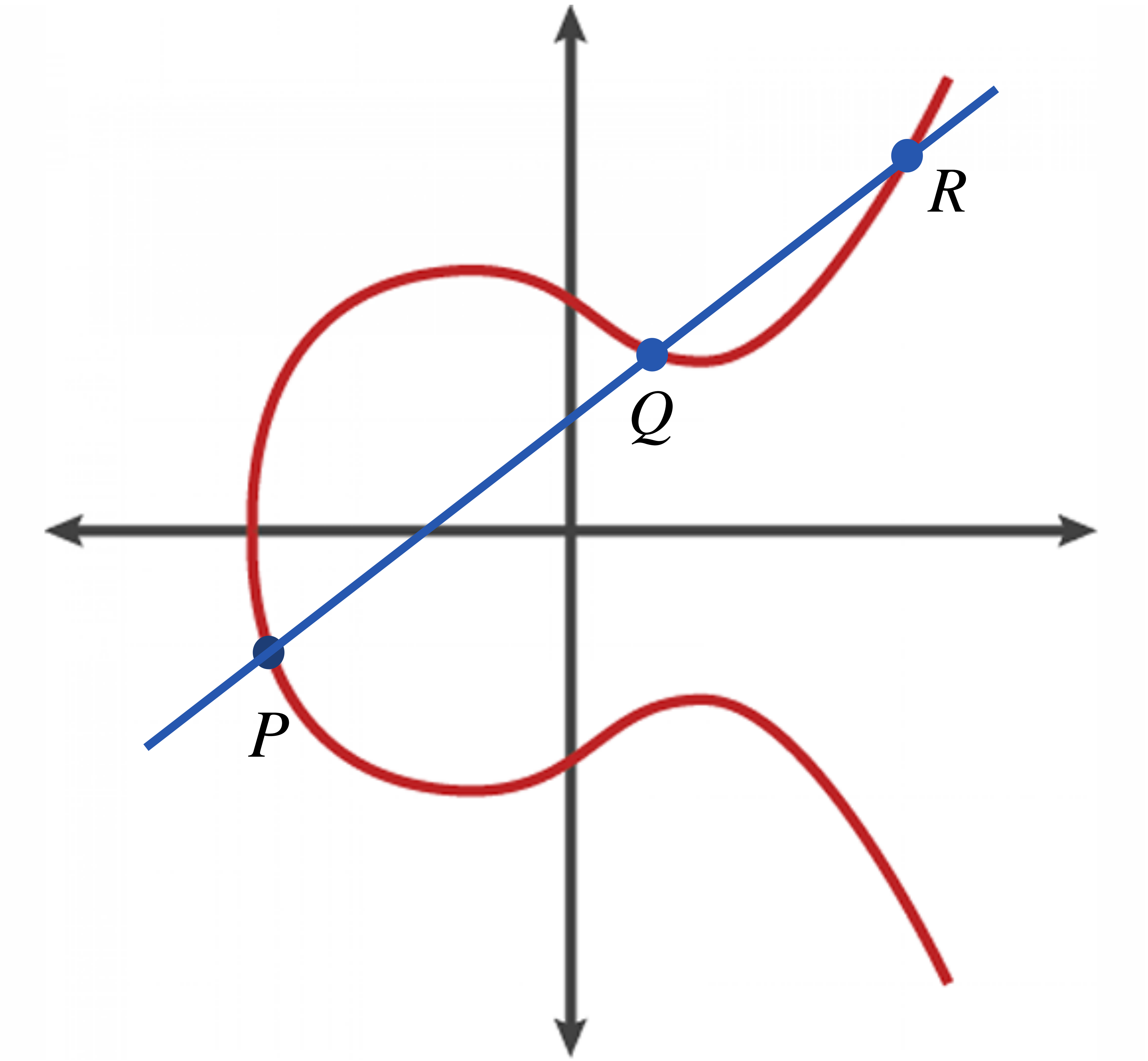

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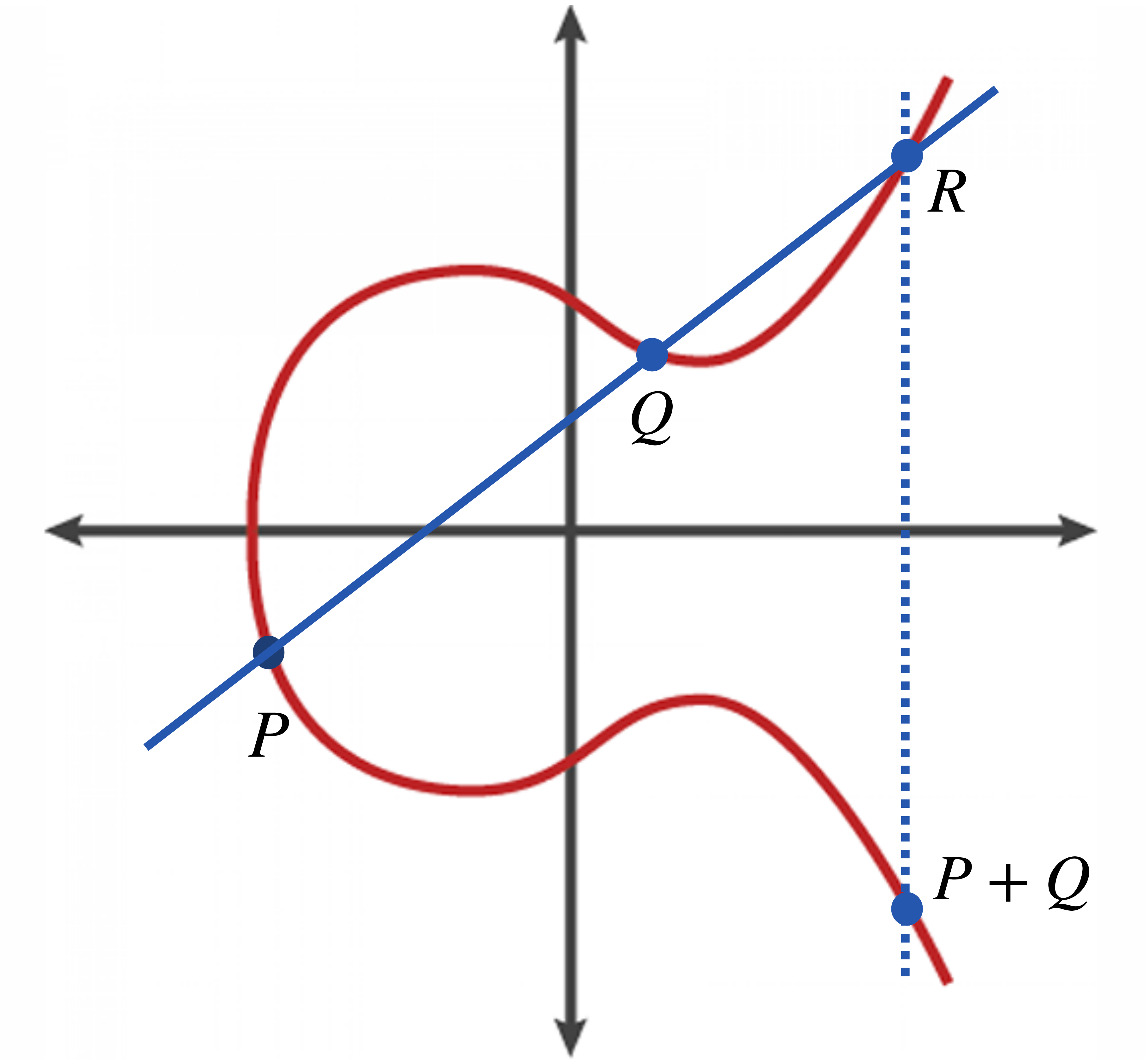

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The **order** of the point is the smallest positive $n \in \mathbb{Z}$ such that $[n]P = \mathcal{O}$.

$E(\mathbb{F}_q)[n]$ is the subgroup of n-torsion points (points whose order divides n).

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An elliptic curve is **supersingular** if $E(\bar{\mathbb{F}}_q)$ has no points of order p, otherwise **ordinary**.

Abelian varieties

Elliptic curves E

Abelian varieties

Elliptic curves E $\longrightarrow$ Principally polarised abelian varieties A

Pairings

A pairing is a non-degenerate bilinear map

(additive) group
of prime order ℓ

$$e : G_1 \times G_2 \to G_T$$

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(multiplicative) group
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Pairings

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$$e : G_1 \times G_2 \to G_T$$

Non-degenerate: $e(P_1, P_2) = 1_{G_T}$ for all $P_2 \in G_2$ if and only if $P_1 = 0_{G_1}$

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Bilinear: For all $P_1, Q_1 \in G_1$ and $P_2, Q_2 \in G_2$ we have

$$e(P_1 + Q_1, P_2) = e(P_1, P_2)e(Q_1, P_2) \text{ and } e(P_1, P_2 + Q_2) = e(P_1, P_2)e(P_1, Q_2)$$

Pairings

For an elliptic curve, the ℓ-Weil pairing (for prime ℓ with ℓ-torsion defined over $\mathbb{F}_{p^u}$) is the pairing

$$e : E(\mathbb{F}_{p^u})[\ell] \times E(\mathbb{F}_{p^u})[\ell] \to \mu_\ell \subseteq \mathbb{F}_{p^{uk}}^\times$$

where k is the *embedding degree with respect to* ℓ: the smallest natural number k such that $\ell \mid (p^u)^k - 1$.

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For **security**, we need the DLP to be hard in $\mathbb{F}_{p^{uk}}^\times$ so we want to maximise the embedding degree (while the pairing still be efficiently computable).

We also need ℓ to be large enough so that the ECDLP is hard in $E(\mathbb{F}_{p^u})[\ell]$; best attack is generic and runs in $O(\sqrt{l})$.

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Paring friendly with respect to q

$$\begin{tikzcd} E/\mathbb{F}_p \arrow[r, bend left] & E'/\mathbb{F}_q \arrow[l, bend left] \end{tikzcd}$$

Paring friendly with respect to p

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We can generalise the ℓ-Weil pairing for principally polarised abelian varieties.

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Embedding Degree

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Embedding Degree $\longrightarrow$

Cryptographic Exponent

For large enough ℓ, the *cryptographic exponent* c_A of A is such that $(p^u)^{c_A} = p^r$, with r the smallest integer such that $\ell \mid p^r - 1$.

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Embedding Degree $\longrightarrow$

When $u > 1$, r can be smaller than uk

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Embedding Degree $\longrightarrow$ **Cryptographic Exponent**

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Captures the ratio between the field of definition of A and where the ℓ-Weil pairing is defined.

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Remark: the larger the cryptographic exponent is, the smaller p^u can be chosen, making arithmetic in the field of definition more efficient while still ensuring DLP security in the finite field group.

The *Ideal* Situation

The ideal situation

A 2-cycle of two prime order elliptic curves, both of which have (the same) embedding degree that balances the ECDLP and DLP securities of all groups involved.

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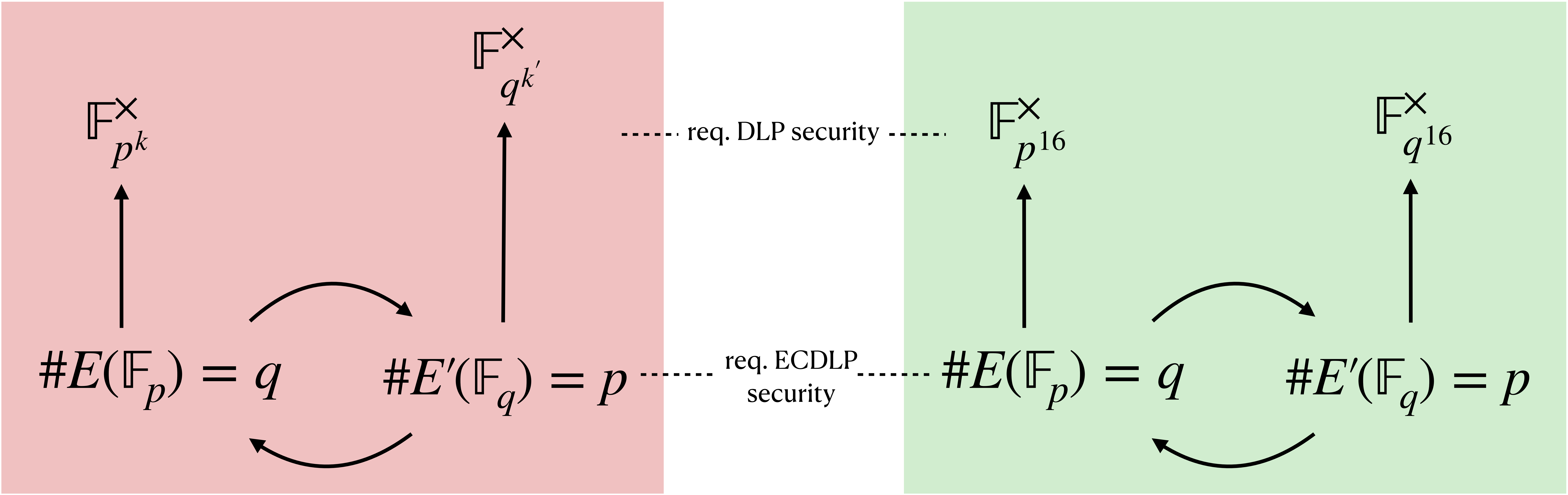

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Security level	**80**	**112**	**128**
Req. ext. field size	1184	3012	3968
Dream cycles $p \approx q$	160	224	256
MNT reality $p \approx q$	296	753	992

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Overview

Our generalisation

We instead consider **pairing-friendly cycles of abelian varieties**. We say

$$A/\mathbb{F}_{p^u} \quad \text{and} \quad B/\mathbb{F}_{q^v}$$

form a cycle if $q \mid \#A(\mathbb{F}_{p^u})$ and $p \mid \#B(\mathbb{F}_{q^v})$ and A, B are pairing-friendly with respect to q, p (respectively).

Differences:

(i) A and B can be abelian varieties of any dimension

(ii) A and B can be defined over extension fields

(iii) p and q need only divide the respective group orders of B and A

High-level strategy

Choices and Restrictions

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To counter: find large v such that v is the smallest integer with $B[p] \subseteq B(\mathbb{F}_{q^v})$

Security

Supersingular?

Security

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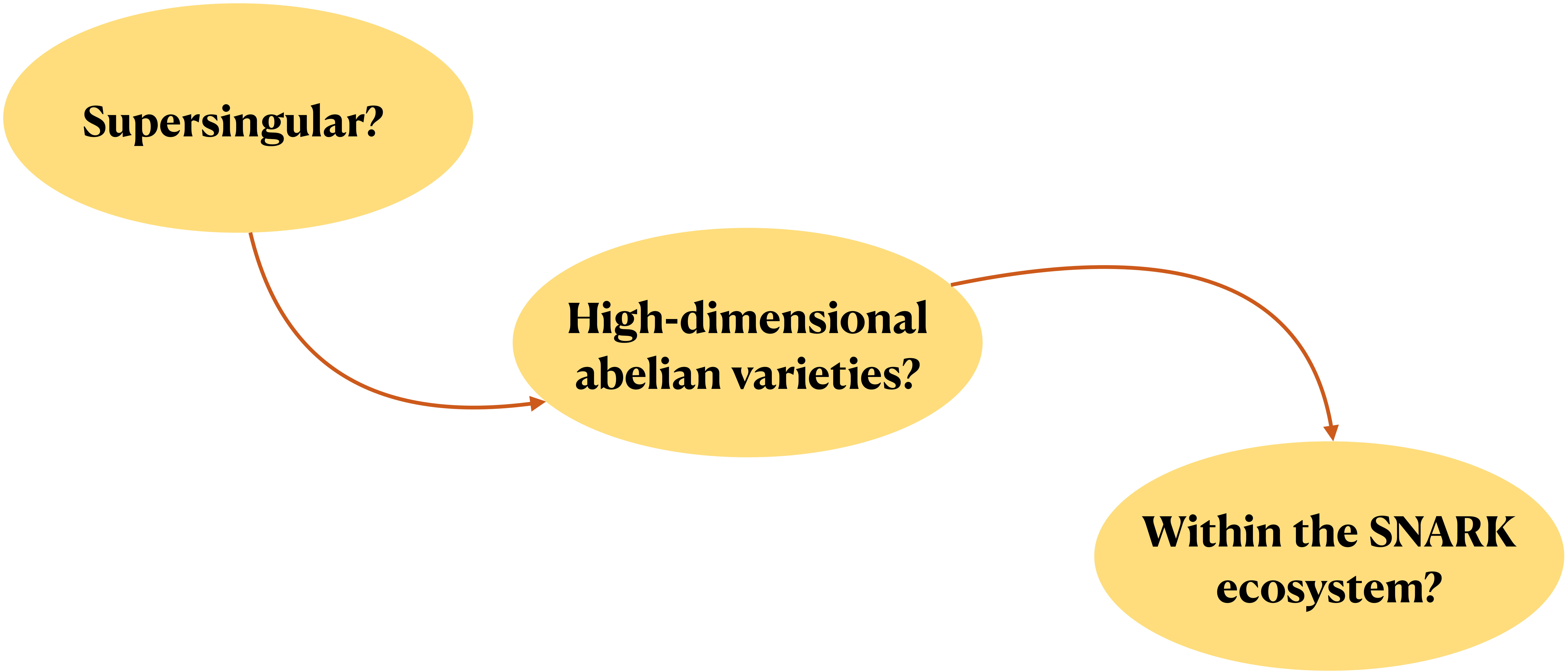
High-dimensional abelian varieties?

Security


Supersingular?
High-dimensional abelian varieties?
Within the SNARK ecosystem?

Towards optimal cycles

Recall: An optimal cycle for λ-bit security would be one where

- $p \approx q \approx 2^{2\lambda}$
- q-Weil pairing of A map into an extension field large enough to achieve λ-bits of security against state-of-the-art DLP attacks.
- Likewise for the p-Weil pairing of B

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This work: Give constructions of A where $p \approx 2^{2\lambda}$ is indeed as small as possible. With our choices and restrictions, we cannot pair this with an "optimal" B… but there is not negative result saying they cannot exist.

Constructions

Constructions

Let's focus on two constructions...

First construction

$p \equiv 2 \bmod 3$, $u = 2u'$ with u' even such that $q = p^u - p^{u/2} + 1$ is prime

- supersingular elliptic curve defined over $\mathbb{F}_{p^u}$
- $\#A(\mathbb{F}_{p^u}) = q$
- cryptographic exponent $c_A = 3$ w.r.t. q

Proposition 2 gives the explicit construction.

$$A/\mathbb{F}_{p^u} \rightleftarrows B/\mathbb{F}_{q^v}$$

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$p > 3$ prime, u even such that $q = p^u + p^{u/2} + 1$ is prime

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- $p \,|\, \#B(\mathbb{F}_{q^2})$
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Proposition 5 gives the explicit construction.

Example #1

$p = 2^{160} - 44159$

$\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ with $\alpha^2 = 3$

$\mathbb{F}_{p^4} = \mathbb{F}_{p^2}(\beta)$ with $\beta^2 = \alpha$

$q = p^4 - p^2 + 1 \approx 2^{640}$

$$\begin{tikzcd}
\mathbb{F}_{p^{12}}^{\times} \ \text{1920-bit field} & \mathbb{F}_{q^2}^{\times} \ \text{1280-bit field} \\
A/\mathbb{F}_{p^4} : y^2 = x^3 + (\alpha + \beta) \arrow[u] \arrow[r, bend left] & B/\mathbb{F}_{q^2} : y^2 = x^3 + ax + b \arrow[u] \arrow[l, bend left]
\end{tikzcd}$$

$a, b \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$

Summary of constructions

Target security	MNT cycle				This work					
	p	q	p^k	$q^{k'}$	$\dim(A)$	p	p^u	q	$(p^u)^{c_A}$	q^v
80	298	298	1192	1788	1	**160**	640	640	1920	1280
112										
128										

Second construction

Let $g = 2^\ell$ with $\ell \geq 0$. Let $u = 2u'$ with u' odd and $q = p^{ug} - p^{ug/2} + 1$ prime.

- supersingular abelian variety over $\mathbb{F}_{p^u}$ of dimension g
- $\#A(\mathbb{F}_{p^u}) = q$
- cryptographic exponent $c_A = 3 \cdot 2^{g-1}$ w.r.t. q

Theorem 4 gives the explicit construction. In particular, there is a natural identification of $A(\mathbb{F}_{p^u})$ with a subgroup of $E(\mathbb{F}_{p^{ur}})$ with $r = 2g$ and E as in Proposition 1. Relies heavily on work by Rubin and Silverberg.

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the trace-zero subvariety

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Proposition 5 gives the explicit construction.

Example #2

$p = 2^{160} - 44159$

$\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ with $\alpha^2 = 3$

$g = 2$

$q = p^4 - p^2 + 1 \approx 2^{640}$

$$
\begin{tikzcd}
\mathbb{F}_{p^{12}}^{\times} \text{ 1920-bit field} & \mathbb{F}_{q^2}^{\times} \text{ 1280-bit field} \\
A/\mathbb{F}_{p^2} \text{ of dimension 2} \arrow[u] \arrow[r, bend left] & \begin{array}{c} B/\mathbb{F}_{q^2} : y^2 = x^3 + ax + b \\ a, b \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q \end{array} \arrow[u] \arrow[l, bend left]
\end{tikzcd}
$$

Summary of constructions

Target security	MNT cycle				This work					
	p	q	p^k	$q^{k'}$	$\dim(A)$	p	p^u	q	$(p^u)^{c_A}$	q^v
80	298	298	1192	1788	1 2	**160** **160**	640 320	640 640	1920 1920	1280 1280

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112	753	753	3012	4517	1 2	**224** **377**	1792 754	1792 1508	5376 4512	3584 3012
128	992	992	3966	5948	1 2 4	2048 1024 **512**	2048 1024 **512**	2048 2048 2048	6144 6144 12288	4096 4096 4096

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The prime numbers p and q have (k, k')-order reciprocity if

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(k, k') will correspond to the embedding degrees.

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- The prime pair $(p, q) = (620461, 15493)$ is such that $(k, k') = (12, 12)$

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- We need higher dimensions:
  - $C/\mathbb{F}_q : y^2 = x^6 + 6611x^5 + 13858x^4 + 6818x^3 + 5652x^2 + 10423x + 1795$ ordinary, $\#J_C(\mathbb{F}_q) = 383 \cdot p$
  - $E/\mathbb{F}_p : y^2 = x^3 + 30984x + 426966$ ordinary, $\#E(\mathbb{F}_p) = 40 \cdot q$

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3. Are there any fixed values of (k, k') with $\min(k, k') > 2$ for which that are no primes with (k, k')-order reciprocity?

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