Elliptic curves for SNARK and proof systems

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Outline

zk-SNARK

Elliptic Curves and Pairings

Proof-friendly curves

Zero-knowledge proofs (ZKP)

Alice

I know the solution to
this complex equation

Examples:
On this chess board, I know mat in 3 moves
I know where is Wally (Charlie) on this drawing
I know a solution to this sudoku grid
I know a preimage of this hash function value

Bob

No idea what the solution is
but Alice claims to know it

Challenge ←

Response →

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- **Sound**: **Alice** has a wrong solution $\Longrightarrow$ **Bob** is not convinced. (valide / validité)

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Challenge ←

Response →

- **Sound**: **Alice** has a wrong solution $\Longrightarrow$ **Bob** is not convinced. (valide / validité)
- **Complete**: **Alice** has the solution $\Longrightarrow$ **Bob** is convinced. (complet / complétude)

Zero-knowledge proofs (ZKP)

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I know the solution to this complex equation

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On this chess board, I know mat in 3 moves
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No idea what the solution is but Alice claims to know it

Challenge ←

Response →

- **Sound**: **Alice** has a wrong solution $\Longrightarrow$ **Bob** is not convinced. (valide / validité)
- **Complete**: **Alice** has the solution $\Longrightarrow$ **Bob** is convinced. (complet / complétude)
- **Zero-knowledge**: **Bob** does NOT learn the solution. (divulgation nulle de connaissance)

Example: Sigma protocol

Alice

Bob

I know $x \in \mathbf{Z}_q$ such that
$g^x = y$ in $\mathbf{G}$, $\#\mathbf{G} = q$ prime

Example: Sigma protocol

Alice | **Bob**

I know $x \in \mathbf{Z}_q$ such that
$g^x = y$ in $\mathbf{G}$, $\#\mathbf{G} = q$ prime

$n \xleftarrow{\$} \mathbf{Z}_q$ $\xrightarrow{A = g^n}$

Example: Sigma protocol

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$n \xleftarrow{\$} \mathbf{Z}_q$

$A = g^n$ →

← c

$c \xleftarrow{\$} \mathbf{Z}_q$

Example: Sigma protocol

Alice		**Bob**
I know $x \in \mathbf{Z}_q$ such that $g^x = y$ in $\mathbf{G}$, $\#\mathbf{G} = q$ prime		
$n \xleftarrow{\$} \mathbf{Z}_q$	$\xrightarrow{A = g^n}$	
	$\xleftarrow{c}$	$c \xleftarrow{\$} \mathbf{Z}_q$
$s = n + c \cdot x \bmod q$	$\xrightarrow{s}$	

Example: Sigma protocol

Alice — **Bob**

I know $x \in \mathbf{Z}_q$ such that $g^x = y$ in $\mathbf{G}$, $\#\mathbf{G} = q$ prime

Alice: $n \xleftarrow{\$} \mathbf{Z}_q$

Alice → Bob: $A = g^n$

Bob: $c \xleftarrow{\$} \mathbf{Z}_q$

Bob → Alice: c

Alice: $s = n + c \cdot x \bmod q$

Alice → Bob: s

Bob: $g^s \stackrel{?}{=} A \cdot y^c$

with $A \cdot y^c = g^n \cdot g^{x \cdot c}$

then $g^n \cdot g^{x \cdot c} = g^{n + x \cdot c}$

Hide the verification
in the exponents
(the scalar field)

Non-Interactive Zero-Knowledge (NIZK) Sigma protocol

Alice **Bob**

I know x such that $g^x = y$

$\mathbf{G}, g, y$

$n \xleftarrow{\$} \mathbf{Z}_q,\ A = g^n$

$c = H(A, y)$

$s = n + c \cdot x \bmod q$

Non-Interactive Zero-Knowledge (NIZK) Sigma protocol

Alice | **Bob**

I know x such that $g^x = y$

$\mathbf{G}, g, y$

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$c = H(A, y)$

$s = n + c \cdot x \bmod q$ $\longrightarrow$ $\pi = (A, c, s)$

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$n \xleftarrow{\$} \mathbf{Z}_q,\ A = g^n$

$c = H(A, y)$

$s = n + c \cdot x \bmod q$

$\xrightarrow{\pi = (A, c, s)}$

$g^s \stackrel{?}{=} A \cdot y^c$

$c \stackrel{?}{=} H(A, y)$

Non-Interactive Zero-Knowledge (NIZK) Sigma protocol

Alice

Bob

I know x such that $g^x = y$

$$\underbrace{\mathbf{G}, g, y}_{\text{Setup}}$$

$$\underbrace{\begin{array}{c} n \xleftarrow{\$} \mathbf{Z}_q,\ A = g^n \\ c = H(A, y) \\ s = n + c \cdot x \bmod q \end{array}}_{\text{Prove}}$$

$$\underbrace{\pi = (A, c, s)}_{\text{proof}} \longrightarrow$$

$$\underbrace{\begin{array}{c} g^s \stackrel{?}{=} A \cdot y^c \\ c \stackrel{?}{=} H(A, y) \end{array}}_{\text{Verify}}$$

zk-SNARK: Zero-Knowledge Succinct Non-interactive ARgument of Knowledge

"I have a *computationally sound*, *complete*, *zero-knowledge*, **succinct**, **non-interactive** proof that a statement is true and that I know a related secret".

Succinct

A proof is very short and easy to verify.

Non-interactive

No interaction between the prover and verifier for proof generation and verification (except the proof message).

ARgument of Knowledge

Honest verifier is convinced that a computationally bounded prover knows a secret information.

zk-SNARKs in a nutshell

Main ideas:

1. Reduce a general statement satisfiability to a polynomial equation satisfiability.
2. Use Schwartz–Zippel lemma to succinctly verify the polynomial equation with high probability.
3. Use homomorphic hiding cryptography to blindly verify the polynomial equation.
4. Make the protocol non-interactive.

Needs of groups for proof systems and SNARK

Statement

group $\mathbf{G}'$ of prime order over $\mathbb{F}_q$ /
Hash function over base field $\mathbb{F}_q$

- `ed_25519` signature verification
$q = 2^{255} - 19$
- Hash function verification $y = H(x)$
H: Poseidon, Anemoi...

Proof

group $\mathbf{G}$ of prime order q over $\mathbb{F}_p$

Group where multiplication
in the exponents is possible:
given g^a, g^b, compute g^{ab}
without knowing a, b
$\rightarrow \approx$ pairing-friendly curves

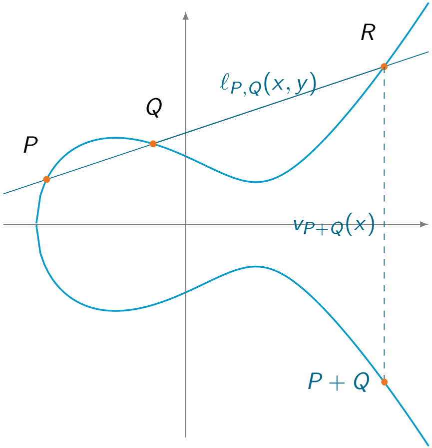
Outline

zk-SNARK

Elliptic Curves and Pairings

Proof-friendly curves

Elliptic curve $E/\mathbb{F}_p$: $y^2 = x^3 + ax + b$, $a, b \in \mathbb{F}_p$, $p \geq 5$, group law



- $E(\mathbb{F}_p)$ has an efficient group law $\rightarrow$ $\mathbf{G}_1$ (chord and tangent rule)
- $\#E(\mathbb{F}_p) = p + 1 - t$, trace t: $|t| \leq 2\sqrt{p}$
- large prime $q \mid p + 1 - t$ coprime to p
- $E(\mathbb{F}_p)[q] = \{P \in E(\mathbb{F}_p) : [q]P = \mathcal{O}\}$ has order q
- $E[q] \simeq \mathbf{Z}/q\mathbf{Z} \times \mathbf{Z}/q\mathbf{Z}$ (for crypto)
- only generic attacks against DLP on well-chosen genus 1 and genus 2 curves
- optimal parameter sizes

Pairing as a black box

$(\mathbf{G}_1, +), (\mathbf{G}_2, +), (\mathbf{G}_T, \cdot)$ three cyclic groups of large prime order q
Pairing: map $e : \mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$

1. bilinear: $e(P_1 + P_2, Q) = e(P_1, Q) \cdot e(P_2, Q)$, $e(P, Q_1 + Q_2) = e(P, Q_1) \cdot e(P, Q_2)$
2. non-degenerate: $e(G_1, G_2) \neq 1$ for $\langle G_1 \rangle = \mathbf{G}_1$, $\langle G_2 \rangle = \mathbf{G}_2$
3. efficiently computable.

Most often used in practice: swap scalars, multiply in the exponents

$$e([a]P, [b]Q) = e([b]P, [a]Q) = e(P, Q)^{ab} \ .$$

Can multiply only once!

$\rightsquigarrow$ Many applications in asymmetric cryptography.

Cryptographic pairing

Modified Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \; : \; E(\mathbb{F}_p)[q] \times E(\mathbb{F}_{p^k})[q] \longrightarrow \mathbb{F}_{p^k}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Cryptographic pairing

Modified Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \; : \; E(\mathbb{F}_p)[q] \times E(\mathbb{F}_{p^k})[q] \longrightarrow \mathbb{F}_{p^k}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Attacks

- inversion of e : hard problem (exponential)
- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{q})$)
- discrete logarithm computation in $\mathbb{F}_{p^k}^*$: **easier, subexponential** $\rightarrow$ take a large enough field

Finding pairing-friendly curves

Designed on purpose: otherwise $k \approx q$
Choose prime integer q, degree k then obtain p: inefficient curve
Design families: parameterized $p(x), q(x), t(x)$

- Complex Multiplication (CM) equation: $t^2 - 4p = -Dy^2$
- (Compute $t^2 - 4p$, get its square-free factorization)
- D discriminant, square-free (in number theory, if $D = 1, 2 \bmod 4$ then $D \leftarrow 4D$)

SEA: from coefficients to parameters

$E/\mathbb{F}_p \colon y^2 = x^3 + ax + b$
Schroof–Elkies–Atkin (SEA)
compute trace t
order $q = p + 1 - t$
change a, b if q not prime

CM: from parameters to coefficients

base field $\mathbb{F}_p$, trace t, order q
CM equation $t^2 - 4p = -Dy^2$
compute Hilbert Class polynomial $H_D(X)$
compute a root $H_D(j) = 0 \bmod p$
$E/\mathbb{F}_p \colon y^2 = x^3 + \frac{3j}{j-1728}x + \frac{2j}{1728-j}$

First ordinary pairing-friendly curves: MNT [MNT01]

Miyaji, Nakabayashi, Takano, $\#E(\mathbb{F}_p) = p(u) + 1 - t(u) = q(u)$

$$k = 3 \begin{cases} t(u) = -1 \pm 6u \\ q(u) = 12u^2 \mp 6u + 1 \\ p(u) = 12u^2 - 1 \\ Dy^2 = 12u^2 \pm 12u - 5 \end{cases}$$

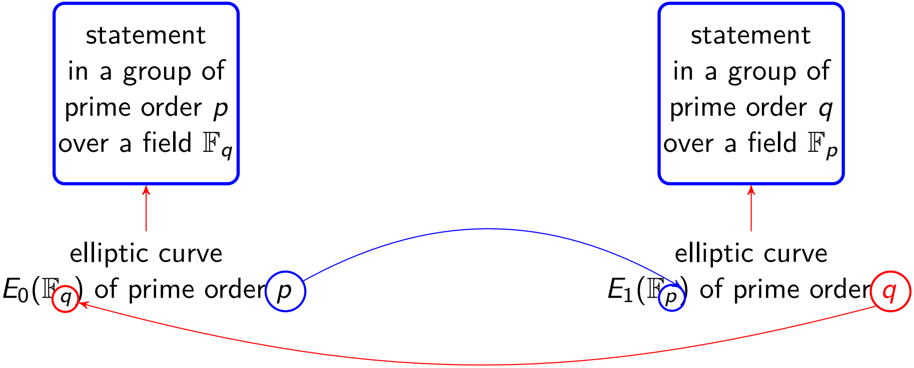
$$k = 4 \begin{cases} t(u) = -u, \ u + 1 \\ q(u) = u^2 + 2u + 2, \ u^2 + 1 \\ p(u) = u^2 + u + 1 \\ Dy^2 = 3u^2 + 4u + 4 \end{cases} \qquad k = 6 \begin{cases} t(u) = 1 \pm 2u \\ q(u) = 4u^2 \mp 2u + 1 \\ p(u) = 4u^2 + 1 \\ Dy^2 = 12u^2 - 4u + 3 \end{cases}$$

CODA [MS18]:

$k = 6$, 753 bits, $E_6 \approx 137$ bits of security, $D = -241873351932854907$, seed $u =$
`0xaa3a58eb20d1fec36e5e772ee6d3ff28c296465f137300399db8a5521e18d33581a262716214583d3b89820dd0c000`

$k = 4$, 753 bits, $E_4 \approx 113$ bits of security

Cycle of curves: unlimited chains of SNARKs [BCTV14]



MNT-4 and MNT-6 curves form a cycle

$k = 4$, MNT-4 parameters	$t_4 = -v$,	$q_4 = v^2 + 1$,	$p_4 = v^2 + v + 1$
$k = 6$, MNT-6 parameters	$t_6 = 1 - 2u$,	$q_6 = 4u^2 + 2u + 1$,	$p_6 = 4u^2 + 1$

$$\begin{matrix} q_4 = p_6 \\ \text{and} \\ p_4 = q_6 \end{matrix} \iff \begin{matrix} v = 2u \\ \text{and} \\ q_4,\ q_6 \text{ are primes} \end{matrix}$$

Unique known cycle of pairing-friendly curves. Impossibility results:

Alessandro Chiesa, Lynn Chua, and Matthew Weidner.
On cycles of pairing-friendly elliptic curves.
SIAM Journal on Applied Algebra and Geometry, 3(2):175–192, 2019.

Marta Bellés-Muñoz, Jorge Jiménez Urroz, and Javier Silva.
Revisiting cycles of pairing-friendly elliptic curves.
In H. Handschuh and A. Lysyanskaya, eds., *CRYPTO 2023, Part II*, vol. 14082 of *LNCS*, pp. 3–37.

New paper with higher genus Abelian varieties:

Maria Corte-Real Santos, Craig Costello, and Michael Naehrig.
On cycles of pairing-friendly abelian varieties.
In L. Reyzin and D. Stebila, eds., *CRYPTO 2024*. ePrint 2024/869.

Very popular pairing-friendly curves: Barreto-Naehrig (BN) [BN06]

$$E_{BN}: \ y^2 = x^3 + b, \ p \equiv 1 \bmod 3, \ D = 3 \text{ (ordinary)}, \ j_E = 0$$

$$\begin{aligned} p &= 36u^4 + 36u^3 + 24u^2 + 6u + 1 \\ t &= 6u^2 + 1 \\ q &= p + 1 - t = 36u^4 + 36u^3 + 18u^2 + 6u + 1 \\ t^2 - 4p &= -3(6u^2 + 4u + 1)^2 \ \rightarrow \ \text{no CM method needed} \end{aligned}$$

Comes from the Aurifeuillean factorization of $\Phi_{12} : \Phi_{12}(6u^2) = q(u)q(-u)$

Security level	$\log_2 q$	$\log_2 p$	k	finite field	$\rho = \log p / \log q$
102	256	256	12	3072	1
123	384	384	12	4608	1
132	448	448	12	5376	1

Formerly BN-254 in Euthereum with seed `0x44e992b44a6909f1`

Barreto, Lynn, Scott curves [BLS03]

Any k, $3 \mid k$, $18 \nmid k$ possible
BLS12 ($k = 12$) becomes more and more popular, replacing BN curves

$$E_{\mathrm{BLS}}: \ y^2 = x^3 + b, \ p \equiv 1 \bmod 3, \ D = 3 \text{ (ordinary)}$$

$$\begin{aligned} p &= (u-1)^2/3(u^4 - u^2 + 1) + u \\ t &= u + 1 \\ q &= (u^4 - u^2 + 1) = \Phi_{12}(u) \\ p + 1 - t &= \underbrace{(u-1)^2/3}_{\text{cofactor}}(u^4 - u^2 + 1) \end{aligned}$$

$$t^2 - 4p = -3y(u)^2 \ \rightarrow \ \text{no CM method needed}$$

BLS12-381 (Zcash [Bow17]) with seed `-0xd201000000010000`
BLS12-377 (Zexe [BCG+]) with seed `0x8508c00000000001`

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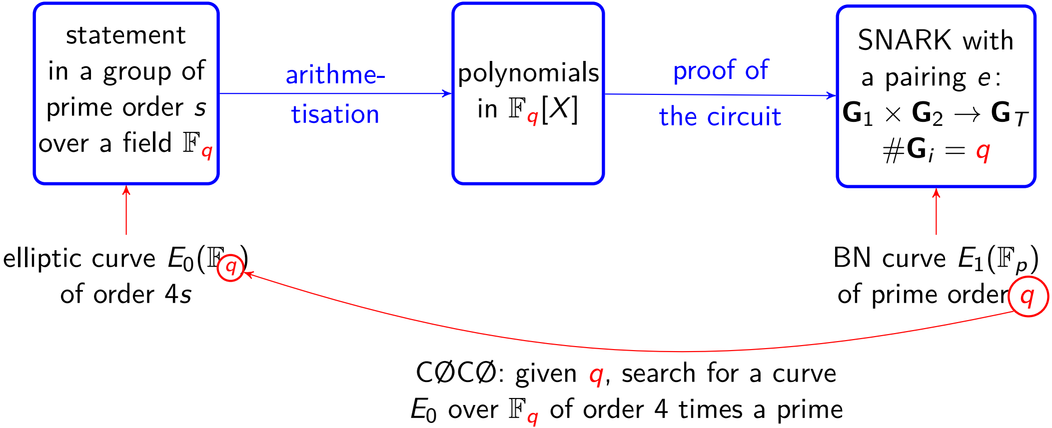
CØCØ embedded curve: Kosba et al. construction [KZM+15]



Embedded SNARK-friendly curves

Usually a twist-secure elliptic curve in Montgomery or (twisted) Edwards form

Input: base field $\mathbb{F}_q$
Output: an embedded curve over $\mathbb{F}_q$ of order $4s$ or $8s$ with prime s
Procedure: Increment the curve coefficients until a suitable curve is found
(Nothing-up-my-sleeves strategy)

CØCØ [KZM+15] with BN-254a,
JubJub [ZCa21] or Bandersnatch [MSZ21] with BLS12-381,
first attempt to generalize Bandersnatch [SEH24]

Bandersnatch [MSZ21]

- Find an embedded elliptic curve E' over $\mathbb{F}_{q_{\text{BLS12-381}}}$ of trace t', above BLS12-381
- With a small discriminant D' in $t'^2 - 4q = -D'y'^2$ to allow faster scalar multiplication with GLV
- twist-secure: $q + 1 - t'$, $q + 1 + t'$ contain a large prime
- Use the CM method

$u = -\texttt{0xd201000000010000}$, $q = u^4 - u^2 + 1$ is prime (BLS12-381)
The trace t' can be any integer in the range $(-2\sqrt{q}; 2\sqrt{q})$
Idea: enumerate small D', get t', order r, twist order r' until r, r' contain a large prime

Bandersnatch curve: $D' = 2$ (i.e. $D' = -8$), $r = 2^2 \times p_{253}$, $r' = 2^7 \cdot 3^3 \times p_{244}$
Is it a *magical* curve? It is *too good to be true*?

More Bandersnatch curves (Joint work with Simon Masson)

Extend the search space for discriminants D'
Rewrite the algorithm to enumerate the curves much faster

Embedded twist-secure curves with BLS12-381:

- $D = 2$, Bandersnatch
- $D = 1030258$, $r = 4p_{253}$, $r' = 2^3 \cdot 7p_{250}$
- $D = 1429201$, $r = 4p_{253}$, $r' = 2^8 \cdot 5p_{245}$
- $D = 1470074$, $r = 2^9 p_{246}$, $r' = 2^2 \cdot 3^4 \cdot 5p_{245}$
- $D = 1992138$, $r = 2^7 p_{248}$, $r' = 2^2 \cdot 3^2 \cdot 79p_{244}$
- $D = 7636102$, $r = 2^2 p_{253}$, $r' = 2^3 \cdot 3^2 \cdot 23p_{245}$
- ...

Embedded prime-order curves with BLS12-381:

- $D = 12387$, r prime
- $D = 6673027$, r prime, $r' = c \cdot p_{234}$ (twist-secure)

Algorithm 1: EmbeddedCurve($q, D_{\min}, D_{\max}$)

Input: prime interger q, minimum and maximum values of $D > 0$
Output: A list of traces and discriminants of embedded elliptic curves for $\mathbb{F}_q$

$\mathcal{L} \leftarrow \{\}$
for D from $D_{\min}$ **to** $D_{\max}$ **do**
  if D is square-free and $-D$ is a square modulo q **then**

$$s \leftarrow \begin{cases} \sqrt{-D} \bmod q & d \not\equiv 3 \bmod 4 \\ \frac{1+\sqrt{-D}}{2} \bmod q & d \equiv 3 \bmod 4 \end{cases}$$

    lift s in $\mathbf{Z}$
    $\pi \leftarrow a + bX$ the shortest non-zero element of the lattice $\mathbb{Z}\langle q, X - s\rangle$
    if π has norm q **then**

$$(t', y') \leftarrow \begin{cases} (2a, b) \text{ **if** } d \equiv 3 \bmod 4 \\ (2a + b, b) \text{ **otherwise**} \end{cases}$$

      if $r = q + 1 - t'$, $r' = q + 1 + t'$ contain a large prime **then**
        $\mathcal{L} \leftarrow \mathcal{L} \cup \{(D, t', y')\}$
return $\mathcal{L}$

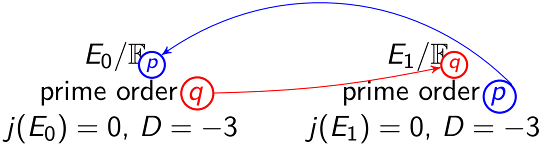
Atkin-Morain, ECPP, and the CM method [AM93]

- internal step in ECPP: find an elliptic curve over $\mathbf{Z}/n\mathbf{Z}$ of non-prime order of known factorization
- enumerate small D until a curve is found
- For each D, solve a norm equation $n = A^2 + DB^2$ in $\mathcal{O}_K$, $K = \mathbf{Q}[\sqrt{-D}]$
- the curve trace is $t' = 2A$, check order
- Do not compute $H_{-D}(X)$ each time, only when a good D is found

More plain/hybrid cycles of curves

Plain cycles: 2 plain prime-order elliptic curves (no pairing)

secp256k1/secq256k1 https://moderncrypto.org/mail-archive/curves/2018/000992.html
HALO: Tweedledum/tweedledee curves https://github.com/daira/tweedle
HALO2: Pallas-Vesta – Pasta curves https://github.com/zcash/pasta_curves



Hybrid cycles: a plain curve and a BN pairing-friendly curve, both prime order

BN254-Grumpkin https://hackmd.io/@aztec-network/ByzgNxBfd
BN382-plain https://github.com/o1-labs/zexe/tree/master/algebra/src/bn_382
Pluto (BN446) - Eris https://github.com/daira/pluto-eris/

ed_25519 as an embedded curve

$q = 2^{255} - 19$

- Curve25519 in Montgomery form

$E' \colon y^2 = x^3 + 48662x^2 + x$

- Ed25519 in twisted Edwards form

$E' \colon -x^2 + y^2 = 1 - \frac{121665}{121666}x^2y^2$

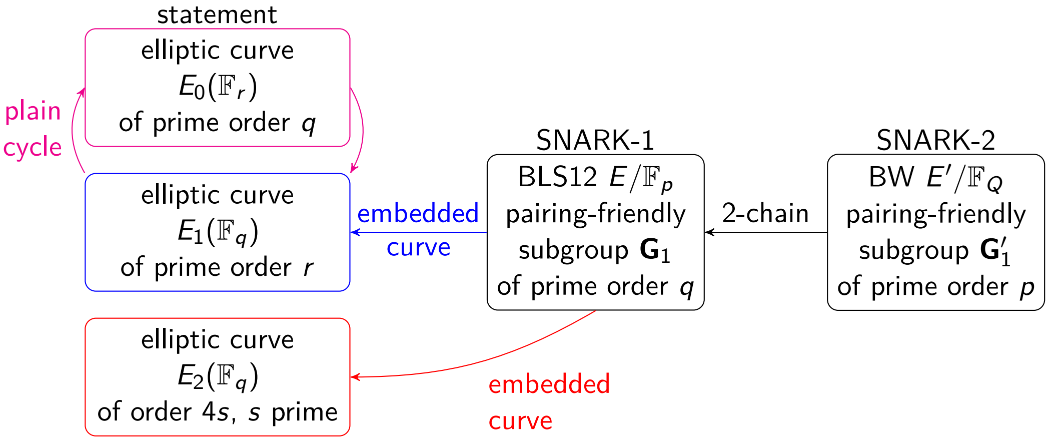
$E'(\mathbb{F}_q)$ of order $8r$, r prime

Prime p, curve $E/\mathbb{F}_p$ of prime order q

- $D = 65012179$
- $D = 103953715$

E' embedded curve of E

Families of embedded curves



Aurore Guillevic.
More embedded curves for snark-pairing-friendly curves.
ePrint:2024/752, 2024.

BLS12 with embedded curves

seed	L	equation $E_{BLS}/\mathbb{F}_p$	p (bits)	q (bits)	embedded curve equation $E_{1,2}/\mathbb{F}_q$	plain cycle curve equation $E_0/\mathbb{F}_r$
`0xffff007fda000001`	25	$y^2 = x^3 + 1$	383	256	$E_1\colon y^2 = x^3 + 19$	$y^2 = x^3 + 7$
$2^{64} - 2^{48} + 2^{39} - 2^{29} - 2^{27} + 2^{25} + 1$					$E_2\colon y^2 = x^3 + 17$	
`0xfc3ec00400000001`	34	$y^2 = x^3 + 1$	383	256	$E_1\colon y^2 = x^3 + 23$	$y^2 = x^3 + 29$
$2^{64} - 2^{58} + 2^{54} - 2^{48} - 2^{46} + 2^{34} + 1$					$E_2\colon y^2 = x^3 + 29$	
`-0xef000ffefdffffff`	25	$y^2 = x^3 + 1$	382	256	$E_1\colon y^2 = x^3 + 11$	$y^2 = x^3 + 17$
$-2^{64} + 2^{60} + 2^{56} - 2^{44} + 2^{32} + 2^{25} + 1$					$E_2\colon y^2 = x^3 + 17$	
`0xdf07fffdfc000001`	26	$y^2 = x^3 + 1$	382	256	$E_1\colon y^2 = x^3 + 11$	$y^2 = x^3 + 7$
$2^{64} - 2^{61} - 2^{56} + 2^{51} - 2^{33} - 2^{26} + 1$					$E_2\colon y^2 = x^3 + 23$	

Some technicalities

- $q(u) = u^4 - u^2 + 1 = \Phi_{12}(u)$ (BLS12),
  $q(u) = (u^6 + 37u^3 + 343)/343$ (KSS18),
  $q(u) = (u^8 + 48u^4 + 625)/61250$ (KSS16)
- Solve for $t'(u), y'(u)$ in $4q(u) = t'(u)^2 + Dy'(u)^2$

Solution:

- Combine Dai–Lin–Zhao–Zhou [DLZZ23] with Smith [Smi15, §4]
- BLS12 [SEH24] $t' = 2u^2 - 1$, $y' = 1$
- KSS16 $t' = (31(u/5)^4 + 1)/7$, $y' = (-17(u/5)^4 - 1)/14$
- KSS18 $t' = -20(u/7)^3 - 1$, $y' = -18(u/7)^3 - 1$
- Consider the quadratic twists, 3rd and 6-th twists ($D = 3$), 4-th twists ($D = 1$)

Our Algorithm

E has endomorphism ϕ, char. poly $\chi(X) = X^2 - t_\phi X + \deg_\phi$
$t_\phi^2 - 4\deg_\phi = -Dn^2$ and $-D$ matches E's in $t^2 - 4p = -Dy^2$

1. $\lambda(x) \leftarrow$ a root of $\chi(X)$ mod $q(x)$
   e.g. if $\chi(X) = X^2 + D$, $\lambda(x) = \sqrt{-D} = (t(x) - 2)/y(x) \bmod q(x)$
2. $U(x), V(x) \leftarrow \text{half-gcd}(q(x), \lambda(x))$
3. with Smith's technique [Smi15, §4], reduce the matrix
   $\begin{bmatrix} U(x) & -V(x) \\ -t_\phi U(x) + \deg_\phi V(x) & U(x) \end{bmatrix}$ whose determinant is

   $\det = U^2 - t_\phi UV + \deg_\phi V^2 = \text{Res}(\chi(X), U - VX)$
   to obtain a short row $(a_0(x), a_1(x))$
4. $(t', y') = (a_0, a_1)$ if $D = 1, 2 \bmod 4$,
   $(t', y') = (2a_0 - a_1, a_1)$ if $D = 3 \bmod 3$.

Example with KSS16

$E_{\text{KSS16}}: y'^2 = x'^3 + ax'$, $j = 1728$, $D = 1$, $\chi = X^2 + 1$

1. $q(x) = (x^8 + 48x^4 + 625)/61250$, $\lambda_\phi = (x^4 + 24)/7 \bmod q(x)$
2. $U, V = (1, -\lambda_\phi) = (1, -(x^4 + 24)/7)$ (no half-gcd needed)
3. $\det \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} = \det \begin{bmatrix} 1 & -(x^4 + 24)/7 \\ (x^4 + 24)/7 & 1 \end{bmatrix} = 1250 q(x)$
4. find integers $(i, j) \bmod 1250 = 2 \cdot 5^4$ such that the denominator simplifies in $(i\boldsymbol{b}_1 + j\boldsymbol{b}_2)/1250 = (i + j(x^4 + 24)/7, i(x^4 + 24)/7 - j)/1250$
5. $x \equiv 25, 45 \bmod 70$ by construction (KSS16) $\implies$ $x \equiv 5 \bmod 10$ $\implies$ $5^4 \mid x^4$. Write $x = 10x_0 + 5 = 5(2x_0 + 1))$ $\implies$ it simplifies to $i + 807j \equiv 0 \bmod 2 \cdot 5^4$.
6. enumerate over (i, j) and keep those such that $(a_0, a_1) = (i\boldsymbol{b}_1 + j\boldsymbol{b}_2)$ satisfies $a_0^2 + a_1^2 = q(x)$

We obtain:

$$\begin{aligned} (i, j) =& (31, 17), \\ (t', y') =& (31\boldsymbol{b}_1 + 17\boldsymbol{b}_2)/1250 = ((17(x/5)^4 + 1)/14, (31(x/5)^4 + 1)/14) . \end{aligned}$$

Embedded curves for KSS16

Parameters (t', y') such that $q = (t'^2 + y'^2)/4$ with $D = 1$.

	(t', y') s.t. $q = (t'^2 + 4y'^2)/4$	$s = q + 1 - t'$	family
t', y'	$(31(u/5)^4 + 1)/7, (-17(u/5)^4 - 1)/7$	$(u^8 - 386u^4 + 5^5 \cdot 17)/61250$	(yes, 2)
$-t', y'$	$(-31(u/5)^4 - 1)/7, (-17(u/5)^4 - 1)/7$	$(u^8 + 482u^4 + 5^4 \cdot 113)/61250$	(yes, 2)
y', t'	$(-17(u/5)^4 - 1)/7, (31(u/5)^4 + 1)/7$	$(u^8 + 286u^4 + 5^4 \cdot 113)/61250$	(yes, 32)
$-y', t'$	$(17(u/5)^4 + 1)/7, (31(u/5)^4 + 1)/7$	$(u^8 - 190u^4 + 5^5 \cdot 17)/61250$	(yes, 20)

Valid seed: $2^{34} - 2^{32} + 2^{30} + 2^{26} - 2^5 - 2^3 - 1$=`0x343fffd7` (row 2), 254-bit order

Thank you for your attention.

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